PROBABILITY STATISTICS

Homework assignment 2 - due January 12$^{\text{th}}$, 2024

Instructions

Please turn in the homework with this cover page. You do not need to edit the solutions. Just make sure the handwriting is legible. You may discuss the problems with your peers but the final solutions should be your work.

Statement: With my signature I confirm that the solutions are the product of my own work. Name: ______________________ Signature: ______________________.

1. Do the following problems from Rice's book:

   Chapter 3: 6, 15, 27, 30, 51, 66.

   Chapter 4: 9, 20, 52, 54, 104.

   Chapter 5: 21, 30.

2. Let $f(x)$ be density of a random variable and let $F(x)$ be its distribution function. Assume $f(x)$ to be piecewise continuous. For some $n \geq 2$ the random variables X and Y have density

$$f_{X,Y}(x,y) = \begin{cases} n(n-1)\left(F(y) - F(x)\right)^{n-2} f(x)f(y) & \text{for } x < y \\ 0 & \text{otherwise.} \end{cases}$$

   a. Compute the densities of random variables X and Y.

   Hint: use

$$\frac{d}{dx}\left[\left(F(y) - F(x)\right)^{n-1}\right] = -(n-1)\left(F(y) - F(x)\right)^{n-2} f(x)\,.$$

   b. Compute the probability

$$P\left(Y - X \geq z\right)$$

   for $z > 0$ in the case where $f(x) = e^{-x}$ for $x > 0$ and $f(x) = 0$ else.

3. There are n white and n black balls in the urn. We are picking the balls randomly without replacement in such a way that the all $\binom{2n}{n}$ arrangements of white and black balls are equally likely.

   a. Compute the probability that just after the selection of $(2k)$-th ball the numbers of white and black balls among the $2k$ balls selected are equal.

   b. Let N be the number of selections just after which the numbers of white and black balls among the selected balls are equal. Show that

$$E(N) = \sum_{k=1}^{n} \frac{\binom{n}{k}\binom{n}{k}}{\binom{2n}{2k}}\,.$$

4. Let $X_1, X_2, \ldots, X_n, X_{n+1}$ be random variables such that $E(X_k) = 0$ for $k = 1, \ldots, n+1$ and covariance matrix $\mathbf{\Sigma}$ (a $(n+1)\times(n+1)$ matrix). We would like to find the best linear predictor for X_{n+1} based on the variables $X_1, X_2, \ldots, X_n$. This means that we are looking for the linear combination $\hat{X}_{n+1} = b_0 + b_1 X_1 + \cdots + b_n X_n$ for which the expected square error

$$E(X_{n+1} - \hat{X}_{n+1})^2$$

will be as small as possible. Find the coefficients $b_0, b_1, \ldots, b_n$.

Hint: Write the square error as a function of $b_0, b_1, \ldots, b_n$ *and use partial derivatives.*

5. Let X and Y be random variables with density

$$f_{X,Y}(x,y) = xe^{-x(y+1)}$$

for $x, y \geq 0$.

   a. Find the conditional densities $f_{X|Y=y}(x)$ and $f_{Y|X=x}(y)$.
   b. Find $E(X|Y)$ and $E(Y|X)$ and check that

$$E(Xg(Y)) = E(E(X|Y)g(Y)) \quad \text{and} \quad E(Yg(X)) = E(E(Y|X)g(X))$$

   for an arbitrary bounded function g.

6. You are offered the following game of chance: from a box containig tickets with numbers on them you select a ticket 1000 times with replacement. If the sum of the numbers on the selected tickets is between a and $a + 100$ inclusive you win. You are free to select a. It is known that the average of the numbers on the tickets is 0.1 and their standard deviation is 1.5811.

   a. Choose $a = 0$. What is the probability that you win?
   b. Choose such an a that your chance of winning will be as large as possible. What is the probability of winning in that case?

7. A magician has two boxes with tickets with numbers on them: the first one with average 1 and standard deviation 10, the second one with average -1 and standard deviation 10. He offers the next game of chance: he will secretly choose one of the boxes, each box with probability 1/2. He will then choose $n = 100$ tickets at random with replacement from the chosen box and will tell us the sum. If we correctly guess the box, we get a price. We decide we will guess in the following way: if the sum is non-negative, we will "guess" the box with average 1, if the sum is negative, we will "guess" the box with average -1.

   a. Assume the magician chooses the box with average 1, but does not tell us that. What is approximate probability that we will guess correctly based on the sum of 100 randomly chosen tickets.

b. Assume again that the magician chooses the box with average 1. How many tickets does he have to choose and tell us the sum to make the probability of a correct guess equal to 0.99?

8. In his 1953 paper with the title *The Random Character of Stock Market Prices* the eminent statistician M. G. Kendall wrote (the quote is edited slightly):

> It seems that the change in price of a stock from one week to the next is practically independent of the change from that week to the week after. This alone is enough to show that it is impossible to predict the price from week to week from the series of weekly prices itself. And if the series really is wandering, any systematic movements such as trends or cycles which may be "observed" in such series are illusory. The series looks like a "wandering" one, almost as if once a week the *Demon of Chance* drew a random number from a large box with average 0, and variance equal to 1 and added it to the current price to determine next week's price. And this, we may recall, is not the behaviour in some small backwater market. The data derive from the Chicago wheat market over a period of fifty years.

a. Suppose we are interested in the net change of price over the next 52 weeks. Fill in the blanks in the following phrase: the net change in price will be just like the ________ of ___ ________ from a box. We assume that the average of the box is ________ and the variance is ________.

b. Find, approximately, the probability that the net change in price over the next 52 weeks will be less than 10.

c. Suppose the *Demon of Chance* decides to change the box. He decides to add the same positive amount on each ticket in the box so that the variance of the box does not change but the average of the box goes up by the amount added. What amount should he add so that the sum of 52 draws will be positive with probability approximately 90%?