University of Ljubljana, Faculty of Economics

Quantitative finance and actuarial science

Probability and statistics

Written examination

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Name and surname: ______________________ ID: | | | | | | | | |

Instructions

Read the problems carefull before staring your work. There are four problems. You can have a sheet with formulae and a mathematical handbook. Write your solutions on the paper provided. You have two hours.

Problem	a.	b.	c.	d.	
1.			•	•	
2.					
3.			•	•	
4.			•	•	
Total					

1. (25) For purposes of sampling the population is divided into K strata of sizes $N_1, N_2, \ldots, N_K$. The sampling procedure is as follows: first a simple random sample of size $k \leq K$ of strata is selected. The selection procedure is independent of the sizes of strata. The second step is then to select a simple random sample in each of the selected strata. If stratum i is selected then we choose a simple random sample of size n_i in this stratum for $i = 1, 2, \ldots, K$. Assume the selection process on the second step is independent of the selection process on the first step.

a. (10) Find an unbiased estimator of the population mean. Explain why it is unbiased.

Hint: let I_i be the indicator that the i-th stratum is selected, and let $\bar{X}_i$ be the sample average for the simple random sample selected in the i-the stratum. The estimator can be written using these random variables. From the description of the sampling procedure we have that the vector $(I_1, I_2, \ldots, I_K)$ is independent of all $\bar{X}_i$, and the variables $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_K$ are independent.

Solution: Define

$$I_i = \begin{cases} 1 & \text{if stratum } i \text{ is chosen,} \\ 0 & \text{else.} \end{cases}$$

From the above it follows that $E(I_i) = P(I_i = 1) = k/K$ for all i. Let $\bar{Y}_i$ be the sample average for the sample chosen in stratum i. We have

$$E(I_i\bar{Y}_i) = E(I_i)E(\bar{Y}_i) = \frac{k}{K} \cdot \mu_i \,.$$

If we put

$$\bar{Y} = \sum_{i=1}^{K} w_i \cdot \frac{K}{k} \cdot I_i\bar{Y}_i$$

we have

$$E(\bar{Y}) = \sum_{i=1}^{K} w_i\mu_i = \mu \,.$$

b. (15) Find the standard error of your unbiased estimator.

Solution: We have

$$\mathrm{var}(\bar{Y}) = \frac{K^2}{k^2}\left[\sum_{i=1}^{K} w_i^2 \mathrm{var}(I_i\bar{Y}_i) + 2\sum_{i<j} w_i w_j \mathrm{cov}(I_i\bar{Y}_i, I_j\bar{Y}_j)\right] .$$

By independence of I_i and $\bar{Y}_i$ we have

$$\mathrm{var}(I_i\bar{Y}_i) = E(I_i)E(\bar{Y}_i^2) - E(I_i)^2E(\bar{Y}_i)^2 \,.$$

We have

$$E(\bar{Y}_i^2) = \mathrm{var}(\bar{Y}_i) + E(\bar{Y}_i)^2 = \frac{\sigma_i^2}{n_i} \cdot \frac{N_i - n_i}{N_i - 1} + \mu_i^2 \,.$$

By independence of (I_i, I_j) *and* $(\bar{Y}_i, \bar{Y}_j)$ *we have*

$$\mathrm{cov}(I_i\bar{Y}_i, I_j\bar{Y}_j) = E(I_iI_j)E(\bar{Y}_i)E(\bar{Y}_j) - \frac{k^2}{K^2}\mu_i\mu_j \,.$$

By definition

$$E(I_iI_j) = P(I_i = 1, I_j = 1) = \frac{k}{K} \cdot \frac{k-1}{K-1} \,.$$

It follows that

$$\mathrm{cov}(I_i\bar{Y}_i, I_j\bar{Y}_j) = \frac{k}{K}\mu_i\mu_j \left(\frac{k-1}{K-1} - \frac{k}{K}\right) \,.$$

Simplifying we find

$$\mathrm{cov}(I_i\bar{Y}_i, I_j\bar{Y}_j) = -\frac{(K-k)k}{(K-1)K^2}\mu_i\mu_j \,.$$

Putting all the pieces together gives the standard error.

2. (20) The Birnbaum-Saunders distribution has the density

$$f(x) = \frac{1}{2\gamma}\left(\frac{1}{x^{1/2}} + \frac{1}{x^{3/2}}\right)\exp\left(-\frac{1}{2\gamma^2}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2\right)$$

for $x > 0$ and $\gamma > 0$. Assume that the observed values $x_1, \ldots, x_n$ are an i.i.d. sample from the density $f(x)$.

a. (5) Find the MLE estimate for the paramater γ.

Solution: The log-likelihood function is

$$\ell(\gamma, \mathbf{x}) = -n\log 2 - n\log\gamma + \sum_{k=1}^{n}\left(\frac{1}{x_k^{1/2}} + \frac{1}{x_k^{3/2}}\right) - \frac{1}{2\gamma^2}\sum_{k=1}^{n}\left(x_k^{1/2} - x_k^{-1/2}\right)^2 .$$

Take the derivative to get

$$\frac{\partial\ell}{\partial\gamma} = -\frac{n}{\gamma} + \frac{1}{\gamma^3}\sum_{k=1}^{n}\left(x_k^{1/2} - x_k^{-1/2}\right)^2 .$$

Set the derivative to zero and solve for γ to get

$$\hat{\gamma} = \sqrt{\frac{1}{n}\sum_{k=1}^{n}\left(x_k^{1/2} - x_k^{-1/2}\right)^2} .$$

b. (5) Assume as known that

$$P(X \leq x) = \Phi\left(\frac{1}{\gamma}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)\right) ,$$

where $\Phi(x)$ is the distribution function of the standard normal distribution. Show that the variable Y defined as

$$Y = \sqrt{X} - \frac{1}{\sqrt{X}}$$

has the $N(0, \gamma^2)$ distribution.

Solution: Denote $f(x) = \sqrt{x} - 1/\sqrt{x}$. The function $f(x)$ is increasing and

$$\begin{aligned} P(Y \leq y) &= P(f(X) \leq y) \\ &= P\left(X \leq f^{-1}(y)\right) \\ &= \Phi\left(\frac{1}{\gamma} f\left(f^{-1}(y)\right)\right) \\ &= \Phi\left(\frac{y}{\gamma}\right) . \end{aligned}$$

c. (5) Is

$$\hat{\gamma}^2 = \frac{1}{n}\sum_{k=1}^{n}\left(\sqrt{X_k} - \frac{1}{\sqrt{X_k}}\right)^2$$

an unbiased estimator of γ^2?

Rešitev: Using part b. compute

$$E\left(\sqrt{X_k} - \frac{1}{\sqrt{X_k}}\right) = \gamma^2 \,.$$

It follows that $\hat{\gamma}^2$ is an unbiased estimate of γ^2.

d. (10) Compute the standard error for $\hat{\gamma}$.

Solution: Compute the second derivative of the log-likelihood function for $n = 1$.

$$\frac{\partial^2 \ell}{\partial \gamma^2} = -\frac{1}{\gamma^2} + \frac{3}{\gamma^4}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) \,.$$

It follows

$$-E\left(\frac{\partial^2 \ell}{\partial \gamma^2}\right) = \frac{2}{\gamma^2} \,.$$

hence

$$\mathrm{se}(\hat{\gamma}) = \frac{\gamma}{\sqrt{2n}} \,.$$

3. (25) Assume the observed values are pairs $(x_1, y_1), \ldots, (x_n, y_n)$. We assume that the pairs are an i.i.d. sample from the bivariate normal density given by

$$f(x,y) = \frac{1}{2\pi\sqrt{ab-c^2}} e^{-\frac{bx^2-2cxy+ay^2}{2(ab-c^2)}}$$

where $a, b > 0$ and $ab - c^2 > 0$. We would like to test the hypothesis

$$H_0\colon c = 0 \qquad \text{versus} \qquad H_1\colon c \neq 0\,.$$

a. (15) Assume as known that the unrestricted maximum likelihood estimates of the parameters are given by

$$\begin{pmatrix} \hat{a} & \hat{c} \\ \hat{c} & \hat{b} \end{pmatrix} = \begin{pmatrix} \frac{1}{n}\sum_{k=1}^n x_k^2 & \frac{1}{n}\sum_{k=1}^n x_k y_k \\ \frac{1}{n}\sum_{k=1}^n x_k y_k & \frac{1}{n}\sum_{k=1}^n y_k^2 \end{pmatrix}$$

Find the likelihood ratio statistic λ for the testing problem.

Solution: The log-likelihood function is given by

$$\ell\,(a,b,c|\mathbf{x},\mathbf{y}) = -n\log 2\pi - \frac{n}{2}\log(ab-c^2) - \frac{1}{2(ab-c^2)}\sum_{k=1}^n (bx_k^2 - 2cx_ky_k + ay_k^2)\,.$$

Using the known unrestricted maximum likelihood estimates we get

$$\ell\left(\hat{a},\hat{b},\hat{c}|\mathbf{x},\mathbf{y}\right) = -n\log 2\pi - \frac{n}{2}\log(\hat{a}\hat{b}-\hat{c}^2) - \frac{1}{2(\hat{a}\hat{b}-\hat{c}^2)}\sum_{k=1}^n (\hat{b}x_k^2 - 2\hat{c}x_ky_k + \hat{a}y_k^2)\,.$$

We need to simplify the last expression. Summing up we get

$$\sum_{k=1}^n (\hat{b}x_k^2 - 2\hat{c}x_ky_k + \hat{a}y_k^2) = \hat{b}n\hat{a} - 2\hat{c}n\hat{c} + \hat{a}n\hat{b}\,.$$

It follows that

$$\ell\left(\hat{a},\hat{b},\hat{c}|\mathbf{x},\mathbf{y}\right) = -n\log 2\pi - \frac{n}{2}\log(\hat{a}\hat{b}-\hat{c}^2) - n\,.$$

In the restricted case we need to maximize

$$\ell\,(a,b|\mathbf{x},\mathbf{y}) = -n\log 2\pi - \frac{n}{2}\log a - \frac{n}{2}\log b - \frac{1}{2a}\sum_{k=1}^n x_k^2 - \frac{1}{2b}\sum_{k=1}^n y_k^2\,.$$

The above expression is maximized when the terms containing a and b are maximized. We get

$$\tilde{a} = \frac{1}{n}\sum_{k=1}^n x_k^2 \quad \text{and} \quad \tilde{b} = \frac{1}{n}\sum_{k=1}^n y_k^2\,.$$

It follows

$$\ell\left(\tilde{a},\tilde{b},0|\mathbf{x},\mathbf{y}\right) = -n\log 2\pi - \frac{n}{2}\log\tilde{a} - \frac{n}{2}\log\tilde{b} - n\,.$$

We have

$$\lambda = n\left(-\log(\hat{a}\hat{b}-\hat{c}^2) + \log\tilde{a} + \log\tilde{b}\right)\,.$$

b. (10) What is the approximate distribution of λ under H_0?

Solution: By Wilks's theorem $\lambda \sim \chi^2(r)$ *where* $r = 3 - 2 = 1$.

4. (25) Assume the following linear regression model:

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

with $E(\boldsymbol{\epsilon}) = 0$ and

$$\mathrm{var}(\boldsymbol{\epsilon}) = \sigma^2 \mathbf{V}\,,$$

where

$$v_{ij} = \frac{\rho^{|i-j|}}{1-\rho^2}\,.$$

Assume that σ^2 is an unknown constant, and $\rho \in (-1, 1)$ is known.

a. (10) Let the components $Z_1, Z_2, \ldots, Z_n$ of the vector $\mathbf{Z}$ be given by the *Cochran-Orcutt* transformation

$$Z_1 = \sqrt{1-\rho^2}\, Y_1 \quad \text{in} \quad Z_i = Y_i - \rho Y_{i-1}$$

for $i = 2, 3, \ldots, n$. Compute $\mathrm{var}(Z_i)$, $\mathrm{cov}(Z_i, Z_j)$ for $i \neq j$.

Solution: Compute

$$\mathrm{var}(Z_1) = \sigma^2\,,$$

and for $i = 2, 3, \ldots n$

$$\begin{aligned}
\mathrm{cov}(Z_1, Z_i) &= \sqrt{1-\rho^2}\,\mathrm{cov}\left(Y_1, Y_i - \rho Y_{i-1}\right) \\
&= \frac{\sigma^2\sqrt{1-\rho^2}}{1-\rho^2}\left(\rho^{i-1} - \rho\cdot\rho^{i-2}\right) \\
&= 0\,.
\end{aligned}$$

Continue to compute $1 < i \leq n$*:*

$$\begin{aligned}
\mathrm{var}(Z_i) &= \mathrm{var}(Y_i - \rho Y_{i-1}) \\
&= \mathrm{var}(Y_i) - 2\rho\,\mathrm{cov}(Y_i, Y_{i-1}) + \rho^2 \mathrm{var}(Y_{i-1}) \\
&= \frac{\sigma^2}{1-\rho^2} - 2\,\frac{\rho^2\sigma}{1-\rho^2} + \frac{\rho^2\sigma^2}{1-\rho^2} \\
&= \sigma^2\,,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{cov}(Z_i, Z_j) &= \mathrm{cov}(Y_i - \rho Y_{i-1}, Y_j - \rho Y_{j-1}) \\
&= \frac{\sigma^2}{1-\rho^2}\left(\rho^{j-i} - \rho^{j-i+2} - \rho^{j-i} + \rho^{j-i+2}\right) \\
&= 0\,.
\end{aligned}$$

b. (15) Find the best unbiased linear estimator of $\boldsymbol{\beta}$.

Solution: Define a new matrix $\tilde{\mathbf{X}}$ *by changing rows* $\mathbf{X}_i$ *of* $\mathbf{X}$ *into*

$$\tilde{\mathbf{X}}_1 = \sqrt{1-\rho^2}\,\mathbf{X}_1 \quad \textit{and} \quad \tilde{\mathbf{X}}_i = \mathbf{X}_i - \rho\mathbf{X}_{i-1}\,.$$

Change the error terms into

$$\eta_1 = \sqrt{1-\rho^2}\,\epsilon_1 \quad \textit{and} \quad \eta_i = \epsilon_i - \rho\epsilon_{i-1}\,.$$

The model

$$\mathbf{Z} = \tilde{\mathbf{X}}\boldsymbol{\beta} + \boldsymbol{\eta}$$

satisfies the assumptions of the Gauss-Markov theorem. The BLUE $\boldsymbol{\beta}$ *is*

$$\hat{\boldsymbol{\beta}} = \left(\tilde{\mathbf{X}}^T\tilde{\mathbf{X}}\right)^{-1}\tilde{\mathbf{X}}^T\mathbf{Z}\,.$$