FAKULTETA ZA MATEMATIKO IN FIZIKO

Oddelek za matematiko

Finančna matematika 2

Pisni izpit

5. junij 2020

Ime in priimek: ______________________________ Vpisna št: | | | | | | | | |

Navodila

Pazljivo preberite besedilo naloge, preden se lotite reševanja. Naloge so 4. Na razpolago imate 2 uri.

Naloga	a.	b.	c.	d.	
1.				•	
2.				•	
3.				•	
4.				•	
Skupaj					

REŠITVE

1. (25) Naj bosta B in D neodvisni standardni Brownovi gibanji. Definiramo

$$T = \inf\{t \geq 0\colon B_t^2 + D_t^2 = 1\}\,.$$

Kot znano predpostavite, da je $E(T^2) < \infty$.

Let B and D be independent standard Brownian motions. We define

$$T = \inf\{t \geq 0\colon B_t^2 + D_t^2 = 1\}\,.$$

As known, assume that $E(T^2) < \infty$.

a. (5) Utemeljite, da je $B_t^2 + D_t^2 - 2t$ martingal in uporabite to za izračun $E(T)$. Utemeljite korake.

Establish that $B_t^2 + D_t^2 - 2t$ is a martingale and use this to compute $E(T)$. Justify your steps.

Solution: $B_t^2 + D_t^2 - 2t$ is a martingale, since it is the sum of two. By optional sampling we have

$$E\left(B_{t\wedge T}^2 + D_{t\wedge T}^2 - 2(T\wedge t)\right) = 0\,.$$

By assumption

$$B_{t\wedge T}^2 + D_{t\wedge T}^2 \to 1$$

as $t \to \infty$ and the sum of squares is bounded by 1. So the expectation converges to 1. On the other hand $t \wedge T \uparrow T$ so by monotone convergence $E(t\wedge T) \to E(T)$. It follows that $E(T) = \frac{1}{2}$.

b. (10) Utemeljite, da je

$$R_t = \left[B_t^2 + D_t^2\right]^2 - 2B_t^2 D_t^2 - 6t(B_t^2 + D_t^2) + 6t^2$$

lokalni martingal.

Verify that

$$R_t = \left[B_t^2 + D_t^2\right]^2 - 2B_t^2 D_t^2 - 6t(B_t^2 + D_t^2) + 6t^2$$

is a local martingale.

Solution: By independence $\langle B, D\rangle = 0$. Denote $Z_t = B_t^2 + D_t^2$. We use Itô to compute

$$\begin{aligned} dR_t &= 4Z_tB_tdB_t + 4Z_tD_tdD_t + 2(3B_t^2 + D_t^2)dt + 2(B_t^2 + 3D_t^2)dt \\ &\quad -4B_tD_t^2dB_t - 4B_t^2D_tdD_t - 2D_t^2dt - 2B_t^2dt \\ &\quad -6Z_tdt - 6t\left(2B_tdB_t + 2D_tdD_t + 2dt\right) + 12tdt \\ &= 4B_t^3dB_t + 4D_t^3dD_t - 6t\left(2B_tdB_t + 2D_tdD_t\right)\,. \end{aligned}$$

Integrals of, say, progressively measurable, locally bounded processes against continuous local martingales are continuous local martingales.

c. (10) Kot znano privzemite, da je

$$E\left(B_T^2 D_T^2\right) = \frac{1}{8}\,.$$

Uporabite dejstvo, da je R lokalni martingal za izračun $E(T^2)$. Utemeljite vaše korake.

Assume as known that

$$E\left(B_T^2 D_T^2\right) = \frac{1}{8}\,.$$

Use the fact that R is a local martingale to compute $E(T^2)$. Justify your steps.

Solution: Note R^T is a bounded continuous local martingale, hence a martingale. Martingales have a constant expectations. Then

$$E\left(R_{t\wedge T}\right) = E(R_0) = 0\,.$$

In $R_{t\wedge T}$ all the terms are dominated by integrable random variables and $R_{t\wedge T}$ coverges to R_T as $t \uparrow \infty$. It follows that

$$0 = 1 - 2E(B_T^2 D_T^2) - 6E(T) + 6E(T^2)\,.$$

Finally we have $E(T^2) = \frac{3}{8}$.

2. (25) Zvezna semimartingala X in Y naj ustrezata sistemu enačb

$$\begin{aligned} dX_t &= tX_t dt + Y_t dt + e^{\frac{t^2}{2}} dW_t^1 \\ dY_t &= tY_t dt + e^{\frac{t^2}{2}} dW_t^2 \end{aligned}$$

kjer sta W^1 in W^2 neodvisni Brownovi gibanji.
Continuous semimartingales X and Y satisfy the equations

$$\begin{aligned} dX_t &= tX_t dt + Y_t dt + e^{\frac{t^2}{2}} dW_t^1 \\ dY_t &= tY_t dt + e^{\frac{t^2}{2}} dW_t^2 \end{aligned}$$

with W^1 and W^2 independent Brownian motions.

a. (10) Pokažite, da sta procesa

$$M_t = e^{-\frac{t^2}{2}} X_t - te^{-\frac{t^2}{2}} Y_t \quad \text{in} \quad N_t = e^{-\frac{t^2}{2}} Y_t$$

martingala.

Show that the processes

$$M_t = e^{-\frac{t^2}{2}} X_t - te^{-\frac{t^2}{2}} Y_t \quad \textit{and} \quad N_t = e^{-\frac{t^2}{2}} Y_t$$

are martingales.

Solution: We compute

$$\begin{aligned} dM_t &= -te^{-\frac{t^2}{2}} X_t dt + e^{-\frac{t^2}{2}} dX_t - (1-t^2) e^{-\frac{t^2}{2}} Y_t dt - te^{-\frac{t^2}{2}} dY_t \\ &= -te^{-\frac{t^2}{2}} X_t dt + e^{-\frac{t^2}{2}} \left(tX_t dt + Y_t dt + e^{\frac{t^2}{2}} dW_t^1 \right) \\ &\quad -(1-t^2) e^{-\frac{t^2}{2}} Y_t dt - te^{-\frac{t^2}{2}} \left(tY_t dt + e^{\frac{t^2}{2}} dW_t^2 \right) \\ &= dW_t^1 - t dW_t^2 \,. \end{aligned}$$

Similarly we get

$$dN_t = dW_t^2 \,.$$

Integrals of locally bounded deterministic functions with respect to Brownian motion are martingales so M and N are martingales.

b. (5) Izračunajte $\langle M, N \rangle_t$.

Compute $\langle M, N \rangle_t$.

Solution: By assumption W^1 and W^2 are independent hence $\langle W^1, W^2 \rangle = 0$. By bilinearity then $\langle M, N \rangle_t = -t^2/2$.

c. (10) Zapišite procesa X in Y eksplicitno pri začetnih pogojih $X_0 = x_0$ in $Y_0 = y_0$.

Given the initial conditions $X_0 = x_0$ and $Y_0 = y_0$ compute the processes X and Y explicitly.

Solution: We have that

$$\begin{aligned} e^{-\frac{t^2}{2}} X_t \quad - \quad t e^{-\frac{t^2}{2}} Y_t &= x_0 + W_t^1 - \int_0^t s dW_s^2 \\ e^{-\frac{t^2}{2}} Y_t &= y_0 + W_t^2 \end{aligned}$$

Solving this linear system of equations we get

$$\begin{aligned} X_t &= e^{\frac{t^2}{2}} (x_0 + t y_0) + e^{\frac{t^2}{2}} W_t^1 + e^{\frac{t^2}{2}} \int_0^t (t - s) \, dW_s^2 \\ Y_t &= e^{\frac{t^2}{2}} y_0 + e^{\frac{t^2}{2}} W_t^2 \end{aligned}$$

3. (25) Naj bosta B^1 in B^2 neodvisni standardni Brownovi gibanji glede na filtracijo $(\mathcal{F}_t)_{t\geq 0}$.
Let B^1 and B^2 be independent Brownian motions relative to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

a. (10) Za $0 \leq t < T$ pokažite, da je

$$\begin{aligned} & E\left(\left(B_T^1 - B_T^2\right)_+ |\mathcal{F}_t\right) \\ = & \sqrt{\frac{T-t}{\pi}} \exp\left(-\frac{(B_t^2 - B_t^1)^2}{4(T-t)}\right) - \left(B_t^2 - B_t^1\right)\left(1 - \Phi\left(\frac{B_t^2 - B_t^1}{\sqrt{2(T-t)}}\right)\right), \end{aligned}$$

kjer je $\Phi(z)$ porazdelitvena funkcija standardizirane normalne porazdelitve. Kot znano upoštevajte, da za neodvisni $Z, W \sim \mathrm{N}(0, T-t)$ velja

$$\begin{aligned} \psi(x,y) &= E\left[((Z-W)-(y-x))_+\right] \\ &= \sqrt{\frac{T-t}{\pi}} e^{-\frac{(y-x)^2}{4(T-t)}} - (y-x)\left(1 - \Phi\left(\frac{y-x}{\sqrt{2(T-t)}}\right)\right). \end{aligned}$$

For $0 \leq t < T$ show that

$$\begin{aligned} & E\left(\left(B_T^1 - B_T^2\right)_+ |\mathcal{F}_t\right) \\ = & \sqrt{\frac{T-t}{\pi}} \exp\left(-\frac{(B_t^2 - B_t^1)^2}{4(T-t)}\right) - \left(B_t^2 - B_t^1\right)\left(1 - \Phi\left(\frac{B_t^2 - B_t^1}{\sqrt{2(T-t)}}\right)\right), \end{aligned}$$

where $\Phi(z)$ is the cumulative distribution function of the standard normal distribution. Take it as known that for independent $Z, W \sim \mathrm{N}(0, T-t)$ one has

$$\begin{aligned} \psi(x,y) &= E\left[((Z-W)-(y-x))_+\right] \\ &= \sqrt{\frac{T-t}{\pi}} e^{-\frac{(y-x)^2}{4(T-t)}} - (y-x)\left(1 - \Phi\left(\frac{y-x}{\sqrt{2(T-t)}}\right)\right). \end{aligned}$$

Solution: We rewrite

$$E\left[\left(B_T^1 - B_T^2\right)_+ |\mathcal{F}_t\right] = E\left[\left(B_T^1 - B_t^1 + B_t^1 - B_T^2 + B_t^2 - B_t^2\right)_+ |\mathcal{F}_t\right].$$

By the simple Markov property of Brownian motion and the lemma on conditioning we have that

$$E\left[\left(B_T^1 - B_T^2\right)_+ |\mathcal{F}_t\right] = \psi(B_t^1, B_t^2)$$

where

$$\psi(x,y) = E\left[(Z + x - W - y)_+\right]$$

with Z, W independent and $Z, W \sim \mathrm{N}(0, T-t)$. We compute

$$
\begin{aligned}
\psi(x,y) &= E\left[((Z-W)-(y-x))_+\right] \\
&= E\left[\sqrt{2(T-t)}\left(\frac{(Z-W)}{\sqrt{2(T-t)}}-\frac{(y-x)}{\sqrt{2(T-t)}}\right)_+\right] \\
&= \sqrt{2(T-t)}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(u-a)_+\,e^{-u^2/2}du \qquad a=\frac{(y-x)}{\sqrt{2(T-t)}} \\
&= \sqrt{2(T-t)}\cdot\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty}(u-a)\,e^{-u^2/2}du \\
&= \sqrt{\frac{T-t}{\pi}}\int_a^\infty ue^{-u^2/2}du - a\sqrt{\frac{2(T-t)}{2\pi}}\int_a^\infty e^{-u^2/2}du \\
&= \sqrt{\frac{T-t}{\pi}}e^{-a^2/2} - a\sqrt{2(T-t)}\,(1-\Phi(a))\ .
\end{aligned}
$$

b. (5) Če lahko zapišemo

$$E\left[\left(B_T^1-B_T^2\right)_+|\mathcal{F}_t\right]=F(B_t^1,B_t^2,t)$$

za $0\leq t<T$ in je funkcija $F(x,y,t)$ dvakrat zvezno parcialno odvedljiva po x in y in enkrat zvezno parcialno odvedljiva po t za $t<T$, potem za $0\leq t<T$ velja

$$
\begin{aligned}
&E\left(\left(B_T^1-B_T^2\right)_+|\mathcal{F}_t\right) \\
&= E\left[\left(B_T^1-B_T^2\right)_+\right]+\int_0^t\frac{\partial F}{\partial x}(B_s^1,B_s^2,s)dB_s^1+\int_0^t\frac{\partial F}{\partial y}(B_s^1,B_s^2,s)dB_s^2\,.
\end{aligned}
$$

Utemeljite zgornjo trditev. Privzamete lahko, da je

$$E\left[\left(B_T^1-B_T^2\right)_+|\mathcal{F}_0\right]=E\left[\left(B_T^1-B_T^2\right)_+\right]\ .$$

If we can write

$$E\left[\left(B_T^1-B_T^2\right)_+|\mathcal{F}_t\right]=F(B_t^1,B_t^2,t)$$

for $0\leq t<T$ and the function $F(x,y,t)$ is twice continuously partially differentiable in x and y and once continuously partially differentiable in t for $t<T$, then for $0\leq t<T$ one has

$$
\begin{aligned}
&E\left(\left(B_T^1-B_T^2\right)_+|\mathcal{F}_t\right) \\
&= E\left[\left(B_T^1-B_T^2\right)_+\right]+\int_0^t\frac{\partial F}{\partial x}(B_s^1,B_s^2,s)dB_s^1+\int_0^t\frac{\partial F}{\partial y}(B_s^1,B_s^2,s)dB_s^2\,.
\end{aligned}
$$

Establish the preceding claim. You may assume that

$$E\left[\left(B_T^1 - B_T^2\right)_+ | \mathcal{F}_0\right] = E\left[\left(B_T^1 - B_T^2\right)_+\right] .$$

Solution: By Itô's formula we have for $0 \le t < T$

$$\begin{aligned} & F(B_t^1, B_t^2, t) - F(B_0^1, B_0^2, 0) \\ = & \int_0^t \frac{\partial F}{\partial x}(B_s^1, B_s^2, s) dB_s^1 + \int_0^t \frac{\partial F}{\partial y}(B_s^1, B_s^2, s) dB_s^2 + \int_0^t \frac{\partial F}{\partial t}(B_s^1, B_s^2, s) ds \\ & + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(B_s^1, B_s^2, s) ds + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial y^2}(B_s^1, B_s^2, s) ds \,. \end{aligned}$$

In the preceding display all the terms are continuous in t*. The left side is a martingale and the first two terms on the right are local martingales. The last three terms have finite total variation so their sum must be constant. Since the sum is* 0 *at* $t = 0$*, the sum vanishes. Furthermore,* $F(0,0,0) = E\left[(B_T^1 - B_T^2)_+\right]$ *by assumption.*

c. (10) Poiščite progresivno merljiva integranda H in K, da bo

$$E\left(\int_0^T H_s^2 ds\right) < \infty \quad \text{in} \quad E\left(\int_0^T K_s^2 ds\right) < \infty$$

ter

$$\left(B_T^1 - B_T^2\right)_+ = E\left[\left(B_T^1 - B_T^2\right)_+\right] + \int_0^T H_s dB_s^1 + \int_0^T K_s dB_s^2 \,.$$

Find progressively measurable integrands H *and* K*, for which*

$$E\left(\int_0^T H_s^2 ds\right) < \infty \quad \textit{and} \quad E\left(\int_0^T K_s^2 ds\right) < \infty$$

and

$$\left(B_T^1 - B_T^2\right)_+ = E\left[\left(B_T^1 - B_T^2\right)_+\right] + \int_0^T H_s dB_s^1 + \int_0^T K_s dB_s^2 \,.$$

Solution: The function $F(x, y, t)$ *from the first part has continuous first and second order partial derivatives for* $t < T$*. We compute*

$$\frac{\partial F}{\partial x}(x, y, t) = \Phi\left(\frac{x - y}{\sqrt{2(T - t)}}\right)$$

and

$$\frac{\partial F}{\partial y}(x, y, t) = -\Phi\left(\frac{x - y}{\sqrt{2(T - t)}}\right)$$

We set

$$H_t = -K_t = \Phi\left(\frac{B_t^1 - B_t^2}{\sqrt{2(T-t)}}\right) .$$

The integrands have limits as $t \uparrow T$ *with probability* 1 *and satisfy the integrability conditions.Take the limit as* $t \uparrow T$ *in part b (using part a), to find that* H *and* K *are the sought after processes.*

4. (25) Naj bo S cena delnice v Black-Scholesovem modelu z obrestno mero r in volatilnostjo $\sigma > 0$. Fiksirajmo zapadlost $T \in (0, \infty)$ in označimo s Q martingalsko mero za interval $[0, T]$, tako da je pod Q na $[0, T]$ diskontiran proces $\tilde{S}$ martingal, W pa Brownovo gibanje. Začetna cena delnice je $S_0 > 0$.

Geometrijska Azijska opcija z izvršilno ceno $K \in (0, \infty)$ in zapadlostjo T izplača

$$Y = \left(S_0 \exp\left(\frac{1}{T} \int_0^T \log(e^{-rt} S_t / S_0) dt \right) - K \right)_+ .$$

ob času T. Kot znano privzemite, da je pod mero Q za $0 \leq t < T$ slučajna spremenljivka $\int_t^T (W_s - W_t) ds$ neodvisna od $\mathcal{F}_t$ in ima $\mathrm{N}\left(0, (T-t)^3/3\right)$ porazdelitev. Kot znano tudi privzemite, da je za $Z \sim \mathrm{N}(a, b^2)$ in $c > 0$

$$E\left[\left(e^Z - c \right)_+ \right] = e^{a + \frac{b^2}{2}} \Phi\left(\frac{a + b^2 - \log c}{b} \right) - c\Phi\left(\frac{-\log c + a}{b} \right),$$

kjer je $\Phi(z)$ porazdelitvena funkcija standardizirano normalne porazdelitve.

Let S be the price of the stock in the Black-Scholes model with interest rate $r \in \mathbb{R}$ and volatility $\sigma \in (0, \infty)$. Fix a maturity $T \in (0, \infty)$ and denote by Q the martingale measure for the interval $[0, T]$, so that under Q on $[0, T]$, the discounted process $\tilde{S}$ is a martingale and W is a Brownian motion. The initial price of the stock is $S_0 > 0$. A geometric Asian option with strike $K \in (0, \infty)$ and maturity T pays out

$$Y = \left(S_0 \exp\left(\frac{1}{T} \int_0^T \ln(S_t e^{-rt} / S_0) dt \right) - K \right)^+$$

at time T. As known, assume that under the measure Q for $0 \leq t < T$ the random variable $\int_t^T (W_s - W_t) ds$ is independent of $\mathcal{F}_t$ and has the $\mathrm{N}\left(0, (T-t)^3/3\right)$ distribution. As known also assume, that for a $Z \sim \mathrm{N}(a, b^2)$ and $c > 0$

$$E\left[\left(e^Z - c \right)_+ \right] = e^{a + \frac{b^2}{2}} \Phi\left(\frac{a + b^2 - \log c}{b} \right) - c\Phi\left(\frac{-\log c + a}{b} \right),$$

with $\Phi(z)$ the cumulative distribution function of the standard normal distribution.

a. (9) Izračunajte začetno vrednost V_0 zgornje opcije.

Compute the initial value V_0 of the above option.

Solution: Since for $t \in [0, T]$, $S_t e^{-rt} / S_0 = \mathcal{E}(\sigma W)_t$, we have

$$V_0 = e^{-rT} Q(S_0 \exp(A_T) - K)^+,$$

where $A_T = \int_0^T (\sigma W_t - \sigma^2 t/2) dt/T \sim_Q N(-\sigma^2 T/4, \sigma^2 T/3) = N(-\sigma^2 T/12 - \sigma^2 T/6, \sigma^2 T/3)$. The analogous expression for the plain vanilla European option, strike K, expiry T, is

$$e^{-rT} Q(S_0 \exp(C_T) - K)^+,$$

with $C_T \sim_Q N(rT - \sigma^2 T/2, \sigma^2 T)$*. It follows that in order to obtain* V_0*, we may simply effect the following changes in the formula for the initial value of the plain vanilla European option (in this order): multiply by* e^{rT}*, send* $\sigma^2 \to \sigma^2/3$*, send* $r \to -\sigma^2/12$*, finally multiply back by* e^{-rT}*. This yields:*

$$V_0 = e^{-rT}(S_0 e^{-\sigma^2 T/12}\Phi(d_1) - K\Phi(d_2))$$

with

$$d_1 = \frac{\log(S_0/K) + \frac{\sigma^2 T}{12}}{\sigma\sqrt{T/3}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T/3}\,.$$

b. (9) Izračunajte vrednostni proces $V = (V_t)_{t\in[0,T]}$ zgornje opcije.

Compute the value process $V = (V_t)_{t\in[0,T]}$ *of the above option.*

Solution: $V_T = Y$*. For* $t < T$

$$V_t = e^{-r(T-t)} Q_{\mathcal{F}_t}\left[(S_0 \exp(A_T) - K)_+\right]$$

and it follows from the lemma on conditioning, the simple Markov property of Brownian motion, part a., and simple rearrangements, that we need only change in the expression from part b. $S_0 \to S_0 e^{\frac{1}{T}\int_0^T W_{s\wedge t} - \frac{\sigma^2 (s\wedge t)}{2} ds}$ *and* $T \to T - t$*.*

c. (7) Kako bi s samo-financirajočim trgovanjem v delnici in denarnem računu replicirali vrednostni proces V? Ni potrebno navesti eksplicitnih formul za poziciji v denarnem računu in delnici, opišite pa kako bi do njih prišli.

By self-financing trading in the stock and in the money market account, how would you replicate the value process V*? You need not provide explicit formulae for the positions in the money market account and the stock, but you should describe how you would obtain them.*

Solution: One sees that $\int_0^T W_s^t ds = W_t(T-t) + \int_0^t W_s ds = W_t T - \int_0^t s dW_s$*.* W *is a Brownian motion under* Q*, and* $M_t = \int_0^t s dW_s$ *is a continuous square-integrable martingale under* Q*. Seeing* V_t *from the preceding part, once multiplied by* e^{-rt} *to obtain the discounted value process, as a suitable function* F *of* W_t*,* M_t *and time* t*, we apply to it Itô's formula. Then, at least on* $[0,T)$*, one would write* $\tilde{V}_t = D_t V_t$*, a martingale, as the sum of* $\tilde{V}_0 = D_0 V_0 = V_0$*, of continuous local martingales (more specifically of integrals against* W *and* M*) and of continuous adapted FV processes. By the usual argument it would follow that the latter vanish. The integration against* M *is, by associativity of stochastic integration, equivalent to integration against* W*. Finally, integration against the latter is (again by associativity of stochastic integration) equivalent to integration against*

$\tilde{S} = DS$. Then such a replication of $\tilde{V} = DV$ by $\tilde{S} = DS$ is in turn equivalent to a self-financing replication of V by trading in the stock and in the money market account. Doing this on $[0, T)$, by the a.s. convergence of $F(\ldots) \to Y$ as $t \uparrow T$, is sufficient.