FACULTY OF MATHEMATICS AND PHYSICS

Department of Mathematics

Financial mathematics 2

Written examination

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Instructions

Read carefully the problems before starting to solve them. There are 4 problems. You have two hours.

Problem	a.	b.	c.	d.	Total
1.					
2.			•	•	
3.			•	•	
4.			•	•	
Total					

1. (25) Assume that B is standard Brownian motion and define the stopping time

$$T = \inf\{t \geq 0 \colon B_t \in \{-1, 3\}\}\,.$$

Assume as known that $P(T < \infty) = 1$. Define the process

$$M_t^\lambda = \exp\left(\lambda(B_t - 1) - \frac{\lambda^2}{2} t\right)$$

for $\lambda \in \mathbb{R}$.

a. (5) Show that for every $\lambda \in \mathbb{R}$ the processes

$$\frac{1}{2}\left(M_t^\lambda + M_t^{-\lambda}\right) = e^{-\frac{\lambda^2 t}{2}} \cosh(\lambda(B_t - 1))$$

and

$$\frac{1}{2}\left(M_t^\lambda - M_t^{-\lambda}\right) = e^{-\frac{\lambda^2 t}{2}} \sinh(\lambda(B_t - 1))$$

are martingales.

Solution: observe that

$$\exp\left(\lambda B_t - \frac{\lambda^2}{2} t\right)$$

is a martingale. Multiplying this martingale by the constant $e^{-\lambda}$ gives a martingale. The two processes are linear combinations of such martingales and hence martingales.

b. (5) Show that

$$E\left(e^{-\frac{\lambda^2}{2} T}\right) = \frac{\cosh(\lambda)}{\cosh(2\lambda)}\,.$$

Justify your steps.

Solution: for fixed t we have

$$E\left(\frac{1}{2}\left(M_{t\wedge T}^\lambda + M_{t\wedge T}^{-\lambda}\right)\right) = \cosh(\lambda)$$

by the optional sampling theorem. The martingale in the parentheses is bounded by the same constant for all t. Moreover, we have

$$\frac{1}{2}\left(M_T^\lambda + M_T^{-\lambda}\right) = e^{-\frac{\lambda^2 T}{2}} \cosh(2\lambda)\,.$$

When $t \to \infty$, by the dominated convergence theorem we have

$$E\left(e^{-\frac{\lambda^2 T}{2}} \cosh(2\lambda)\right) = \cosh(\lambda)\,.$$

The result follows.

c. (5) Derive that

$$E\left(e^{-\frac{\lambda^2}{2}T}\cdot 1\left(B_T=3\right)\right)+E\left(e^{-\frac{\lambda^2}{2}T}\cdot 1\left(B_T=-1\right)\right)=\frac{\cosh(\lambda)}{\cosh(2\lambda)}$$

and

$$E\left(e^{-\frac{\lambda^2}{2}T}\cdot 1\left(B_T=3\right)\right)-E\left(e^{-\frac{\lambda^2}{2}T}\cdot 1\left(B_T=-1\right)\right)=-\frac{\sinh(\lambda)}{\sinh(2\lambda)}\,.$$

Justify your steps.

Solution: the first equality is just the point b. and $1\left(B_T=3\right)+1\left(B_T=-1\right)=1$. *For the second, we justify the application of the optional sampling theorem for the martingale*

$$\frac{1}{2}\left(M_t^{\lambda}-M_t^{-\lambda}\right)=e^{-\frac{\lambda^2 t}{2}}\sinh(\lambda(B_t-1))\,.$$

The justification is identical to that for the first martingale.

d. (10) Compute

$$E\left(e^{-\frac{\lambda^2}{2}T}\cdot 1\left(B_T=3\right)\right)\quad\text{and}\quad E\left(e^{-\frac{\lambda^2}{2}T}\cdot 1\left(B_T=-1\right)\right)$$

Solution: in point c. we have a 2×2 *system of linear equations for the two expectations. Solving the system gives*

$$E\left(e^{-\frac{\lambda^2}{2}T}\cdot 1\left(B_T=3\right)\right)=\frac{\sinh(\lambda)}{\sinh(4\lambda)}$$

and

$$E\left(e^{-\frac{\lambda^2}{2}T}\cdot 1\left(B_T=-1\right)\right)=\frac{\sinh(3\lambda)}{\sinh(4\lambda)}\,.$$

2. (25) Let the semimartingale X satisfy the stochastic differential equation

$$dX_t = (1 + X_t)dt + (1 + X_t)dB_t$$

where B is Brownian motion and $X_0 = 1$. Let

$$Z_t = e^{B_t + \frac{1}{2}t}\,.$$

a. (10) Compute $d(X_t Z_t^{-1})$.

Solution: we have

$$Z_t^{-1} = e^{-B_t - \frac{1}{2}t}\,,$$

and by Itô's formula

$$d\left(Z_t^{-1}\right) = -Z_t^{-1} dB_t\,.$$

We compute

$$\begin{aligned} & d(X_t Z_t^{-1}) \\ & = X_t\left(-Z_t^{-1} dB_t\right) + Z_t^{-1}\left((1 + X_t)dt + (1 + X_t)dB_t\right) - Z_t^{-1}(1 + X_t)dt \\ & = Z_t^{-1} dB_t\,. \end{aligned}$$

b. (15) Find a solution of the stochastic differential equation X that does not involve integrals.

Solution: in principle we have

$$X_t Z_t^{-1} = 1 + \int_0^t Z_s^{-1} dB_s\,.$$

However, from the first part we have

$$d\left(Z_t^{-1}\right) = -Z_t^{-1} dB_t\,.$$

In other words,

$$\int_0^t Z_s^{-1} dB_s = -Z_t^{-1} + Z_0^{-1}\,.$$

Combining the two results, we have that

$$X_t = Z_t(2 - Z_t^{-1})\,.$$

Finally,

$$X_t = 2Z_t - 1\,.$$

3. (25) Assume as known that $Z \sim \mathrm{N}(0,1)$ and $z \in \mathbb{R}$

$$E\left(|z+Z|\right) = \sqrt{\frac{2}{\pi}}e^{-z^2/2} + z\mathrm{erf}\left(\frac{z}{\sqrt{2}}\right),$$

where

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}ds$$

is the error function. Let B be standard Brownian motion, and $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration.

a. (10) For $0 \leq t < T$ compute

$$E\left(|B_T|\big|\mathcal{F}_t\right).$$

Solution: we write

$$E\left(|B_T|\big|\mathcal{F}_t\right) = E\left(|B_T - B_t + B_t|\big|\mathcal{F}_t\right).$$

The random variable $B_T - B_t$ is independent of $\mathcal{F}_t$. We have

$$E\left(|B_T|\big|\mathcal{F}_t\right) = \psi(B_t),$$

where

$$\begin{aligned}
\psi(x) &= E\left(|B_T - B_t + x|\right) \\
&= E\left(|\sqrt{T-t}Z + x|\right) \\
&= \sqrt{T-t}E\left(\left|\frac{x}{\sqrt{T-t}} + Z\right|\right) \\
&= \sqrt{\frac{2(T-t)}{\pi}}e^{-\frac{x^2}{2(T-t)}} + x\mathrm{erf}\left(\frac{x}{\sqrt{2(T-t)}}\right).
\end{aligned}$$

b. (15) Find the integrand H, such that

$$|B_T| = E\left(|B_T|\right) + \int_0^T H_s dB_s.$$

Justify your steps.

Solution: if we can write

$$E\left(|B_T|\big|\mathcal{F}_t\right) = F(B_t, t)$$

for $0 \leq t < T$ for a smooth function $F(x,t)$, then

$$|B_T| = E\left(|B_T|\right) + \int_0^T \frac{\partial F}{\partial x}(B_s, s)\, dB_s.$$

It follows that

$$H_t = -\sqrt{\frac{2}{\pi(T-t)}} B_t e^{-\frac{B_t^2}{2(T-t)}} + \operatorname{erf}\left(\frac{B_t}{\sqrt{2(T-t)}}\right) + B_t \sqrt{\frac{2}{\pi(T-t)}} e^{-\frac{B_t^2}{2(T-t)}} .$$

The result simplifies to

$$H_t = \operatorname{erf}\left(\frac{B_t}{\sqrt{2(T-t)}}\right) .$$

Strictly speaking, the above is valid for $t < T$. But taking left limits justifies the result in general.

4. (25) Let S be the price of the stock in the Black-Scholes model with interest rate $r \in \mathbb{R}$ and volatility $\sigma \in (0, \infty)$. Fix a maturity $T \in (0, \infty)$ and the initial price of the stock $S_0 > 0$. The geometric Asian option with strike $K \in (0, \infty)$ and maturity T pays

$$V_T = \left(S_0 \exp\left(\frac{1}{T} \int_0^T \log(S_t e^{-rt}/S_0) dt \right) - K \right)_+$$

at time T. Assume as known that for a $Z \sim \mathrm{N}(a, b^2)$ and $c > 0$

$$E\left[\left(e^Z - c\right)_+ \right] = e^{a+\frac{b^2}{2}} \Phi\left(\frac{a + b^2 - \log c}{b} \right) - c\Phi\left(\frac{-\log c + a}{b} \right),$$

with $\Phi(z)$ the cumulative distribution function of the standard normal distribution. Further, assume as known that for Brownian motion B we have

$$\int_0^t B_s ds \sim \mathrm{N}\left(0, \frac{t^3}{3}\right).$$

a. (10) Compute the initial value V_0 of the above option.

Solution: for $t \in [0, T]$ under Q, we have

$$\log\left(S_t e^{-rt}/S_0\right) = \sigma \tilde{B}_t - \frac{\sigma^2}{2} t\,.$$

We have

$$V_0 = e^{-rT} E_Q\left[(S_0 e^{A_T} - K)_+ \right] = e^{-rT} E_Q\left[(e^{A_T + \log S_0} - K)_+ \right]$$

where

$$A_T = \frac{1}{T} \int_0^T (\sigma \tilde{B}_t - \sigma^2 t/2) dt\,.$$

Under Q, we have that $A_T \sim N(-\sigma^2 T/4, \sigma^2 T/3)$. Using the known formula above, we get

$$V_0 = e^{-rT}(S_0 e^{-\sigma^2 T/12} \Phi(d_1) - K\Phi(d_2))$$

with

$$d_1 = \frac{\log\left(S_0/K\right) + \frac{\sigma^2 T}{12}}{\sigma\sqrt{T/3}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T/3}\,.$$

b. (15) Compute the value process V_t of the above option.

Solution: fix $t \in [0,T]$. *Write*

$$\begin{aligned} A^1 &= \frac{1}{T}\int_0^t \left(\sigma \tilde{B}_s - \frac{\sigma^2}{2}s\right) ds \\ A^2 &= \frac{1}{T}\int_t^T \left(\sigma(\tilde{B}_s - \tilde{B}_t) - \frac{\sigma^2}{2}(s-t)\right) ds \\ A^3 &= \frac{\sigma(T-t)}{T}\tilde{B}_t - \frac{\sigma^2(T-t)}{2T}t \end{aligned}$$

We have $A^1, A^3 \in \mathcal{F}_t$, *and* A^2 *is independent of* $\mathcal{F}_t$. *We need to find*

$$V_t = e^{-r(T-t)} E_Q\left(V_T|\mathcal{F}_t\right) .$$

Rewrite

$$E_Q\left(V_T|\mathcal{F}_t\right) = E_Q\left(\left(S_0 e^{A^1} e^{A^3} e^{A_2} - K\right)_+ |\mathcal{F}_t\right) .$$

We have that

$$A^2 \sim \mathrm{N}\left(-\frac{\sigma^2(T-t)^2}{4T}, \frac{\sigma^2(T-t)^3}{3T^2}\right) .$$

Using independence and the rules for conditional expectations, we can assume that A^1 *and* A^3 *are constants, and we use formula given in the text with*

$$a = \log S_0 + A^1 + A^3 - \frac{\sigma^2(T-t)^2}{4T} \quad \text{and} \quad b = \frac{\sigma^2(T-t)^3}{3T^2}$$

and $c = K$.