## Time series

Time series are statistical models for many types of data which exhibit dependency over time. Let us examine the AR(1) model. Assume that $\epsilon_1, \epsilon_2, \ldots$ are independent and identically distributed as $N(0, \sigma^2)$. Let $X_0 \sim N(0, \tau^2)$ be independent from $\epsilon_1, \epsilon_2, \ldots$ where

$$\tau^2 = \frac{\sigma^2}{1-\rho^2}\,,$$

for $|\rho| < 1$. Let the random variables $X_0, X_1, \ldots$ have the distribution defined by

$$X_n = \rho X_{n-1} + \epsilon_n$$

for $n = 1, 2, \ldots$.

a. Write down the density of $(X_0, X_1, \ldots, X_n)$. Show that all the variables $X_n$ have the same expectation and the same variance.

b. Suppose that the data is created as the random variables $X_0, \ldots, X_n$. Show that the log-likelihood function is

$$\begin{aligned}\ell(\rho, \tau^2 | \mathbf{x}) &= \\ = & -\frac{n+1}{2}\log(2\pi) - \frac{1}{2}\log(\tau^2) - \frac{x_0^2}{2\tau^2} \\ & -\frac{n}{2}\log(\sigma^2) - \sum_{k=1}^{n}\frac{(x_k - \rho x_{k-1})^2}{2\sigma^2}\,.\end{aligned}$$

c. How would you estimate the two parameters $\rho$ and $\sigma^2$?

d. Choose $\rho = 1/2$ in $\sigma = 0, 1$ and simulate the time series. Repeat $N = 1000$-times. Look at the histograms for the estimates obtained by simulation. Comment the histograms.

e. How would you explain the approximate normality of the sampling distributions?