Mixed effects model

Let $\mathbf{Y} = \mathbf{X}\beta + \epsilon$ be a linear model where we assume $E(\epsilon) = \mathbf{0}$ in $\mathrm{var}(\epsilon) = \sigma^2\mathbf{\Sigma}$ for a known invertible matrix $\mathbf{\Sigma}$.

a. Show that the BLUE for β is given by

$$\hat{\beta} = (\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{Y}\,.$$

Assume that $\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X}$ is invertible and use the Gauss-Markov theorem.

b. Assume that the linear model is of the form

$$Y_{kl} = \alpha + \beta x_{kl} + u_k + \epsilon_{kl},$$

$k = 1, 2, \ldots, K$ in $l = 1, 2, \ldots, L_k$ where ϵ_{kl} are $N(0, \sigma^2)$ and u_k are $N(0, \tau^2)$ and all random quantities are independent. Assume that the ratio τ^2/σ^2 is known. Show that the BLUE is given by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = \begin{pmatrix} \sum w_k & \sum w_k \bar{x}_k \\ \sum w_k \bar{x}_k & S_{xx} + \sum w_k \bar{x}_k^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum w_k \bar{y}_k \\ S_{xy} + \sum w_k \bar{x}_k \bar{y}_k \end{pmatrix},$$

where

$$\begin{aligned} w_k &= L_k\sigma^2/(\sigma^2 + L_k\tau^2) \\ S_{xx} &= \sum_k \sum_l (x_{kl} - \bar{x}_k)^2 \\ S_{xy} &= \sum_k \sum_l (x_{kl} - \bar{x}_k)(y_{kl} - \bar{y}_k)\,. \end{aligned}$$

Hint: For $c \neq -1/n$ one has $(\mathbf{I} + c\mathbf{1}\mathbf{1}')^{-1} = \mathbf{I} - c(1 + nc)^{-1}\mathbf{1}\mathbf{1}'$ where $\mathbf{1} = (1, 1, \ldots, 1)$.

c. What would you do if the ratio τ^2/σ^2 were unknown?

d. How would you test the hypothesis $H_0\colon \beta = 0$ versus $H_1\colon \beta \neq 0$? What is the distribution of the test statistic under the null-hypothesis?