Correlation test for multivariate normal distribution

Assume that the data $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are an i.i.d. sample from the multivariate normal distribution of the form

$$\mathbf{X}_1 \sim \mathrm{N}\left(\begin{pmatrix} \boldsymbol{\mu}^{(1)} \\ \boldsymbol{\mu}^{(2)} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}\right) .$$

Assume that the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are unknown. Assume the following theorem:

If $\mathbf{A}(p \times p)$ is a given symmetric positive definite matrix then the positive definite matrix $\boldsymbol{\Sigma}$ that maximizes the expression

$$\frac{1}{\det(\boldsymbol{\Sigma})^{n/2}} \cdot \exp\left(-\frac{1}{2}\mathrm{Tr}\left(\boldsymbol{\Sigma}^{-1}\mathbf{A}\right)\right)$$

is the matrix

$$\boldsymbol{\Sigma} = \frac{1}{n}\mathbf{A} .$$

The testing problem is

$$H_0\colon \boldsymbol{\Sigma}_{12} = 0 \qquad \text{versus} \qquad H_1\colon \boldsymbol{\Sigma}_{12} \neq 0 .$$

a. Find the maximum likelihood estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ in the unconstrained case.

b. Find the maximum likelihood estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ in the constrained case.

c. Write the likelihood ratio statistic for the testing problem as explicitly as possible.

d. What can you say about the distribution of the likelihood ratio statistic?