Three parameter gamma distribution

Define the incomplete gamma function for $x > 0$ as

$$\Gamma(a; x) = \frac{1}{\Gamma(a)} \int_0^x t^{a-1} e^{-t} \, \mathrm{d}t$$

and the derivative of the log-gamma function as

$$\psi(x) = \frac{\mathrm{d} \log \Gamma(x)}{\mathrm{d}x} = \frac{\Gamma'(x)}{\Gamma(x)} .$$

Let X be a random variable with the distribution function

$$F_X(x) = \Gamma(a; (\lambda x)^\tau) ,$$

where $a > 0$, $\lambda > 0$ and $\tau > 0$ are parameters. Denote $X \sim \mathrm{T}\Gamma(a, \lambda, \tau)$.

The unknown parameters will be estimated using the maximum likelihood method. Assume the sample is generated as independent i.i.d. random variables $X_1, \dots, X_n$. The log-likelihood function is

$$\begin{aligned} \ell(a, \lambda, \tau) &= a \tau n \log \lambda + n \log \tau - n \log \Gamma(a) + \\ &\qquad + (a\tau - 1) \sum_{i=1}^n \log x_i - \lambda^\tau \sum_{i=1}^n x_i^\tau \\ &= n \left(a \tau \log \lambda + \log \tau - \log \Gamma(a) + (a \tau - 1) \overline{\log x} - \lambda^\tau \, \overline{x^\tau} \right). \end{aligned}$$

a. Show that the MLE satisfy the equations

$$\frac{1}{n} \frac{\partial \ell}{\partial \lambda} = \frac{a \tau}{\lambda} - \tau \lambda^{\tau - 1} \overline{x^\tau} = 0 ,$$

hence

$$\lambda = \left(\frac{\overline{x^\tau}}{a} \right)^{-\frac{1}{\tau}}$$

and

$$\log \lambda = -\frac{1}{\tau} \log \overline{x^\tau} + \frac{1}{\tau} \log a .$$

b. Show that

$$\frac{1}{n} \frac{\partial \ell}{\partial a} = \tau \log \lambda - \psi(a) + \tau \overline{\log x} = 0$$

and

$$\psi(a) - \log a - \tau \overline{\log x} + \log \overline{x^\tau} = 0.$$

c. From the partial derivative with respect to τ derive that

$$a = \frac{\overline{x^\tau}}{\tau\left(\overline{x^\tau \log x} - \overline{x^\tau}\,\overline{\log x}\right)}.$$

d. Denote the right side of the equation in c. by $g(\tau)$. Show that

$$\psi(g(\tau)) - \log(g(\tau)) - \tau\,\overline{\log x} + \log \overline{x^\tau} = 0\,,$$

which is an equation that only contains τ. Generate a sample of size $n = 1000$ from the generalized gamma distribution and plot the graph of the left side of the above equation. What can you say about the uniqueness of the solution?

e. Compute the Fisher matrix of information $I(a, \lambda, \tau)$.

f. Generate i.i.d. samples of size $n = 1000$ and estimate the parameters. Repeat the procedure $m = 10000$ times. Draw the histograms for the estimates and compute the empirical standard errors. Compare the empirical standard errors with the ones obtained from the Fisher information matrix.

g. How would you test the hypothesis $H_0\colon \tau = 1$ vs. $H_1\colon \tau \neq 1$? Generate the distribution of the test statistic when H_0 holds. Comment.