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Suppose $\{p(\mathbf{x},\theta), \theta \in \boldsymbol{\Theta} \subset \mathbb{R}^k\}$ is a (regular) family of distributions. Define the vector valued *score function* $\mathbf{s}$ as the column vector with components

$$\mathbf{s}(\mathbf{x},\theta) = \frac{\partial}{\partial\theta} \log(p(\mathbf{x},\theta)) = \text{grad}(\log(p(\mathbf{x},\theta)) \,.$$

and the Fisher information matrix as

$$\mathbf{I}(\theta) = \text{var}(\mathbf{s}) \,.$$

Remark: If $p(\mathbf{x},\theta) = 0$ *define* $\log\left(p(\mathbf{X},\theta)\right) = 0$.

a. Let $\mathbf{t}(\mathbf{X})$ be an unbiased estimator of θ based on the likelihood function, i.e.

$$E_\theta(\mathbf{t}(\mathbf{X})) = \theta \,.$$

Prove that

$$E(\mathbf{s}) = \mathbf{0} \qquad \text{and} \qquad E(\mathbf{s}\mathbf{t}^T) = \mathbf{I} \,.$$

Deduce that $\text{cov}(\mathbf{s},\mathbf{t}) = \mathbf{I}$.

Remark: Make liberal assumptions about interchanging integration and differentiation.

b. Let $\mathbf{a},\mathbf{c}$ be two arbitrary k–dimensional vectors. Prove that

$$\text{corr}^2\left(\mathbf{a}^T\mathbf{t}, \mathbf{c}^T\mathbf{s}\right) = \frac{(\mathbf{a}^T\mathbf{c})^2}{\mathbf{a}^T\text{var}(\mathbf{t})\mathbf{a}\cdot\mathbf{c}^T\mathbf{I}(\theta)\mathbf{c}} \,.$$

The correlation coefficient squared is always less or equal 1. Maximize the expression for the correlation coefficient over $\mathbf{c}$ and deduce the Rao-Cramér inequality.