Parameter estimation for the Weibull distribution

The Weibull distribution with parameters α and σ is given by the density

$$f(x, \alpha, \sigma) = \frac{\alpha}{\sigma} \cdot \left(\frac{x}{\sigma}\right)^{\alpha-1} \exp(\,-(x/\sigma)^{\alpha})$$

for $x > 0$ where $\alpha > 0$ and $\sigma > 0$.

a. Let $x_1, x_2, \ldots, x_n$ be the observations with the assumption that they were "produced" from a Weibull distribution. Find the equations that the MLE must satisfy.

b. Show that the MLE are unique except in the case when all observations are equal.

   Hint: Look at the left and right side of the second equation. Show that the right side is a strictly increasing function of α.

c. On p. 264-265 in the Rice (see references) one has

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} \mathbf{N}(0, \mathbf{\Sigma})\,,$$

   where $\mathbf{\Sigma}$ is the asymptotic covariance. Compute this covariance.

   Hin: Note that $\int_0^\infty (\log u)^k u^{p-1} e^{-u} \mathrm{d}u = \Gamma^{(k)}(p)$. You may find it useful to note that the random variable $(X/\sigma)^{\alpha}$ has exponential distribution.

d. Simulate samples of size $n = 1000$ and estimate the standard error for the parameter α. Compare this standard error with the one given by MLE theory. Comment.