University of Primorska

FAMNIT, Mathematics

Probability

Exam

June 3rd, 2020

Name and surname: ______________________ Identification number: | | | | | | | | |

Instructions

Read carefully the text of the problems before attempting to solve them. Five problems out of six count for 100%. You are allowed one A4 sheet with formulae and theorems. You have two hours.

Problem	a.	b.	c.	d.	
1.			•	•	
2.			•	•	
3.			•	•	
4.			•	•	
5.			•	•	
6.			•	•	
Total					

1. (20) The standard deck of cards contains 52 cards of 13 ranks (aces, kings, queens,....). There are 4 cards of each rank. In the game *Baccarat* the dealer takes 8 standard decks of cards and shuffles tham well. We assume that all permutations of the 416 cards are equally likely. The dealer then takes 52 cards from the top of the 416 shuffled cards.

Let A be the event that cards of all 13 ranks are among the 52 top cards. Let B be the event that there is at least one ace among the top 52 cards.

a. (10) What is the probability of A? You do not need to simplify binomial symbols.

Hint: Look at A^c and write it as an union.

Solution: Let A be the event we are interested in and let

$$A_i = \{\textit{the kind } i \textit{ is missing}\}$$

for $i = 1, 2, \ldots, 13$. *We have numbered the kinds. We have* $A^c = \cup_{i=1}^{13} A_i$. *The 52 cards the dealer selects are a random selection of 52 cards out of 416 cards. We calculate*

$$P(A_1) = \frac{\binom{416-32}{52}}{\binom{416}{52}}, \qquad P(A_1 \cap A_2) = \frac{\binom{416-64}{52}}{\binom{416}{52}}$$

and in general

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = \frac{\binom{416-32k}{52}}{\binom{416}{52}}$$

for $k = 1, 2, \ldots, 11$*; for* $k = 12, 13$ *the event* $A_1 \cap A_2 \cap \cdots \cap A_k$ *cannot happen. By symmetry and the inclusion-exclusion*

$$P(A^c) = \sum_{k=1}^{11} (-1)^{k-1} \binom{13}{k} \cdot \frac{\binom{416-32k}{52}}{\binom{416}{52}},$$

which means

$$P(A) = \sum_{k=0}^{11} (-1)^{k} \binom{13}{k} \cdot \frac{\binom{416-32k}{52}}{\binom{416}{52}}.$$

A numeric calculation gives

$$P(A^c) = 0.8554971\,.$$

b. (10) Compute $P(A|B)$.

Solution: We note that $A \subseteq B$ *so*

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{1 - P(B^c)} = \frac{P(A)}{1 - \frac{\binom{416-32}{52}}{\binom{416}{52}}} = \frac{P(A)}{1 - \frac{\binom{384}{52}}{\binom{416}{52}}}\,.$$

The numerical value is

$$P(A|B) \doteq 0{,}865574\,.$$

2. (20) In an urn there are $B \geq 2$ black balls and R red balls. We select balls from the urn one by one randomly without replacement. Let X be the number of balls drawn up to and including the first black ball and let Y be the number of balls drawn up to and including the seconf black ball.

a. (10) Find the joint distribution of X and Y.

Solution: The possible pairs of values of X and Y are (k, l) for which $1 \leq k < l \leq R+2$. The event $\{X = k, Y = l\}$ happens if we first get $k-1$ red balls, a black ball, then $l-k-1$ red balls and finally a black ball. Denote $N = B + R$. The probability of the above event is is

$$
\begin{aligned}
P(X = k, Y = l) &= \frac{R}{N} \cdot \frac{R-1}{N-1} \cdots \frac{R-k+2}{N-k+2} \cdot \frac{B}{N-k+1} \cdot \\
&\qquad \cdot \frac{R-k+1}{N-k} \cdots \frac{R-l+3}{N-l+2} \cdot \frac{B-1}{N-l+1} \\
&= \frac{B(B-1)\, R!\, (N-l)!}{(R-l+2)!\, N!}
\end{aligned}
$$

or

$$
P(X = k, Y = l) = \frac{\binom{N-l}{B-2}}{\binom{N}{B}} = \frac{B(B-1)\, R!\, (N-l)!}{(R-l+2)!\, N!}\,.
$$

b. (10) Show that for $l = 2, 3, \ldots, R+2$ and $k = 1, 2, \ldots, l-1$ we have

$$
P(X = k, Y = l) = \frac{1}{l-1}\, P(Y = l)\,.
$$

Solution: By the formula for marginal distributions we have

$$
P(Y = l) = \sum_{k=1}^{l-1} P(X = k, Y = l)
$$

Note that all the terms in the sum are equal. The assertion follows.

3. (20) Let the random variable T_a have the density

$$f_a(t) = \frac{a}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}}$$

for $a > 0$ and $t > 0$.

a. (10) Suppose T_a and Z are independent with $Z \sim \mathrm{N}(0,1)$. Find the density of $W = \sqrt{T_a} Z$.

Solution: Define

$$\Phi(t,z) = (t, \sqrt{t}z) \,.$$

We have

$$\Phi^{-1}(t,w) = \left(t, w/\sqrt{t}\right) \qquad \text{and} \qquad J_{\Phi^{-1}}(t,w) = \frac{1}{\sqrt{t}} \,.$$

The pair (T_a, W) has the density

$$f_{T_a,W}(t,w) = f_a(t) f_Z\left(\frac{w}{\sqrt{t}}\right) \cdot \frac{1}{\sqrt{t}} \,.$$

The density of W is computed using the marginal density formula. We get

$$\begin{aligned} f_W(w) &= \frac{a}{2\pi} \int_0^\infty \frac{1}{t^2} e^{-\frac{a^2}{2t}} e^{-\frac{w^2}{2t}} \, dt \\ &= \frac{a}{2\pi} \int_0^\infty e^{-\frac{(a^2+w^2)v}{2}} dv \\ &= \frac{a}{\pi(a^2+w^2)} \,. \end{aligned}$$

b. (10) Compute the density of

$$Y = \frac{1}{1 + T_1 Z^2} \,.$$

Solution: The random variable Y will have values on $(0,1)$ with probability 1.

For $y \in (0,1)$ *compute*

$$\begin{aligned} P(Y \leq y) &= P\left(\frac{1}{1+T_1Z^2} \leq y\right) \\ &= P\left(T_1Z^2 \geq \frac{1}{y} - 1\right) \\ &= P\left(\sqrt{T_1}|Z| \geq \sqrt{\frac{1}{y} - 1}\right) \\ &= 2P\left(\sqrt{T_1}Z \geq \sqrt{\frac{1}{y} - 1}\right) \\ &= 1 - \frac{1}{\pi}\mathrm{arctg}\left(\sqrt{\frac{1}{y} - 1}\right) . \end{aligned}$$

Differentiate to get

$$\begin{aligned} f_Y(y) &= \frac{2}{\pi\left(1+\frac{1-y}{y}\right)} \cdot \frac{1}{2\sqrt{1-y} \cdot y^{3/2}} \\ &= \frac{1}{\pi\sqrt{y(1-y)}} . \end{aligned}$$

In other words $Y \sim \mathrm{Beta}(\frac{1}{2}, \frac{1}{2})$.

4. (20) Let $X_1 = 1$ and assume that for $n = 2, 3, \ldots$

$$P\left(X_{n+1} = k | X_n = k\right) = \frac{n+1-k}{n+1} \quad \text{and} \quad P\left(X_{n+1} = k+1 | X_n = k\right) = \frac{k}{n+1}$$

for $k = 1, \ldots, n$. Let

$$Y_n = \frac{X_n(X_n+1)}{(n+1)(n+2)}\,.$$

a. (10) Compute

$$E\left(Y_{n+1} | X_n = k\right)\,.$$

Solution: By definition

$$\begin{aligned}
&E\left(\frac{X_{n+1}(X_{n+1}+1)}{(n+2)(n+3)} | X_n = k\right) \\
&= \frac{1}{(n+2)(n+3)}\left(k(k+1)\cdot\frac{n+1-k}{n+1} + (k+1)(k+2)\frac{k}{n+1}\right) \\
&= \frac{1}{(n+1)(n+2)(n+3)}\cdot k(k+1)((n+1-k)+(k+2)) \\
&= \frac{k(k+1)}{(n+1)(n+2)}\,.
\end{aligned}$$

b. (10) Compute $E\left(X_n(X_n+1)\right)$ for all $n = 1, 2, \ldots,$

Solution: By the formula for total expectation we have

$$\begin{aligned}
E(Y_{n+1}) &= \sum_{k=1}^{n} E(Y_{n+1} | X_n = k) P(X_n = k) \\
&= \sum_{k=1}^{n} \frac{k(k+1)}{(n+1)(n+2)} P(X_n = k) \\
&= E(Y_n)\,.
\end{aligned}$$

We note that $E(Y_1) = 1/3$. *It follows that*

$$E\left(X_n(X_n+1)\right) = \frac{(n+1)(n+2)}{3}\,.$$

5. (20) Let N be a non-negative integer valued random variable such that

$$P(N = n) = \left(a + \frac{b}{n}\right) P(N = n - 1)\,; \quad n = 1, 2, 3, \ldots\,,$$

where $a \in [0, 1)$ and $b \geq 0$. The generating function of N satisfies

$$(1 - as)G'_N(s) = (a + b)G_N(s)\,.$$

This relation also uniquely determines G_N.

a. (10) Compute $E(N)$ and $\mathrm{var}(N)$.

Solution: We have

$$\lim_{s\uparrow 1}(1 - as)G'_N(s) = \lim_{s\uparrow 1}(a + b)G_N(s)\,.$$

By continuity of $G_N(s)$ on $[-1, 1]$ we have

$$\lim_{s\uparrow 1}(a + b)G_N(s) = (a + b)G_N(1) = (a + b)\,.$$

On the left we have

$$\lim_{s\uparrow 1}(1 - as)G'_N(s) = (1 - a)\lim_{s\uparrow 1} G'_N(s) = (1 - a)E(X)\,.$$

It follows that

$$E(N) = \frac{a + b}{1 - a}\,.$$

Differentiating the relation G_N satisfies we get for $s \in (-1, 1)$

$$-aG'_N(s) - (1 - as)G''_N(s) = (a + b)G'_N(s)\,.$$

Taking limits as $s \uparrow 1$ we get

$$-aE(N) + (1 - a)E\left[N(N - 1)\right] = (a + b)E(N)$$

Rearrange to get

$$-E(N) + (1 - a)E(N^2) = (a + b)E(N)\,.$$

We have

$$E(N^2) = \frac{(1 + a + b)E(N)}{1 - a}\,.$$

It follows

$$\mathrm{var}(N) = \frac{(1 + a + b)(a + b)}{(1 - a)^2} - \frac{(a + b)^2}{(1 - a)^2}\,. = \frac{a + b}{(1 - a)^2}\,.$$

b. (10) Let $I_1, I_2, \ldots$ be independent equally distributed random variables with $I_1 \sim \text{Bernoulli}(p)$. Let $X = I_1 + \cdots + I_N$. For X we have

$$P(X = n) = \left(\tilde{a} + \frac{\tilde{b}}{n}\right) P(X = n-1)\,; \quad n = 1, 2, 3, \ldots\,,$$

for $\tilde{a}$ and $\tilde{b}$. Compute $\tilde{a}$ and $\tilde{b}$.

Solution: We know that $G_X(s) = G_N(G_{I_1}(s))$. By definition $G_{I_1}(s) = q + ps$ where $q = 1 - p$. We have

$$G_X(s) = G_N(q + ps)\,.$$

Differentiate to get

$$G_X'(s) = p\, G_N'(q + ps)\,,$$

hence

$$\frac{G_X'(s)}{G_X(s)} = p\,\frac{G_N'(q+ps)}{G_N(q+ps)} = p\,\frac{a+b}{1 - aq - aps} = \frac{\frac{ap}{1-aq} + \frac{bp}{1-aq}}{1 - \frac{aps}{1-aq}}\,.$$

From the equation we infer $\tilde{a} = \frac{ap}{1-aq}$ and $\tilde{b} = \frac{bp}{1-bq}$. We have used uniqueness.

6. (20) You are offered the following game of chance: from a box containig tickets with numbers on them you select a ticket 1000 times with replacement. If the sum of the numbers on the selected tickets is between a and $a+100$ you win. You are free to select a. It is known that the average of the numbers on the tickets is 0.1 and their standard deviation is 1.5811.

a. (10) Choose $a=0$. What is the probability that you win? Use that $\Phi(-2) = 0.0228$.

Solution: We use the central limit theorem to get S_{1000}. *We compute*

$$\begin{aligned} P(0 \leq S_n \leq 100) &= P(-100 \leq S_{1000} - 100 \leq 0) \\ &= P(-\frac{100}{\sqrt{1000} \cdot 1,5811} \leq \frac{S_{1000} - 100}{\sqrt{1000} \cdot 1,5811} \leq 0) \\ &\approx P(-2 \leq Z \leq 0) \\ &= 0,48\,. \end{aligned}$$

b. (10) Choose such an a that your chance of winning will be as large as possible. What is the probability of winning in that case? Use $\Phi(1) = 1 - \Phi(-1) = 0,8413$.

Solution: The histogram for S_{1000} *will be similar to a normal density nd symmetric around* $E(S_{1000}) = 100$. *To get the largest probability we need to choose an interval symmetric around* 100 *which means from* 50 *to* 150. *We estimate*

$$\begin{aligned} &P(50 \leq S_n \leq 150) \\ &= P(-50 \leq S_{1000} - 100 \leq 50) \\ &= P(-\frac{50}{\sqrt{1000} \cdot 1,5811} \leq \frac{S_{1000} - 100}{\sqrt{1000} \cdot 1,5811} \leq \frac{50}{\sqrt{1000} \cdot 1,5811}) \\ &\approx P(-1 \leq Z \leq 1) \\ &= 0,68\,. \end{aligned}$$