University of Primorska

FAMNIT

Probability

Written examination

June 13$^{\text{th}}$, 2018

Name and surname: ____________________ Identification number: | | | | | | | | |

Instructions

Read carefully the text of the problems before attempting to solve them. Five problems out of six count for 100%. You are allowed one A4 sheet with formulae and theorems. You have two hours.

Naloga	a.	b.	c.	d.	
1.			•	•	
2.			•	•	
3.			•	•	
4.			•	•	
5.			•	•	
6.			•	•	
Skupaj					

1. (20) *Craps* is a popular game in Las Vegas. The game has two steps: the gambler rolls two dice first. If the sum of dots on both dice is 7 or 11, the gambler wins. If the sum is 2, 3 or 12, the gambler loses. If the sum of dots is none of the above numbers, it becomes the gambler's "magical number". The gambler continues rolling the two dice. If the sum of 7 appears before the sum equal to the magical number, the gambler loses. If the sum equal to the magical number appears before the sum of 7 the gambler wins.

a. (10) Denote by H_k the event where in first roll shows the sum of k and let A be the event that the gambler wins at the end. Assume as known that

$$P(A|H_k) = \frac{6P(H_k)}{1 + 6P(H_k)}$$

for $k \notin \{2, 3, 7, 11, 12\}$. Compute $P(A)$.

Hint: Use symmetry so that you do not need to sum up many fractions.

b. (10) Suppose that the gamler has won. What is the conditional probability that in the first roll showed a sum which was an even number?

2. (20) For testing random number generators the *Poker test* is used. The following problem occurs: $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5$ are independent, equally distributed random variables with uniform distribution on the set $\{0, 1, \ldots, m-1\}$ for some $m > 0$. Let X denote the number of distinct numbers in the set $\{\xi_1, \xi_2, \xi_3, \xi_4, \xi_5\}$. Example: if the set is $\{1, 2, 5, 2, 5\}$, there are three distinct numbers.

a. (10) Compute $E(X)$.

   Hint: indicators.

b. (10) Compute $\mathrm{var}(X)$.

   Hint: justify that

$$\begin{aligned}
& P(\{\textit{numbers } k \textit{ and } l \textit{ occur}\}) \\
&\quad = \; 1 - P\left(\{k \textit{ does not occur}\} \cup \{l \textit{ does not occur}\}\right) \\
&\quad = \; 1 - P(\{k \textit{ does not occur}\}) - P(\{l \textit{ does not occur}\}) \\
&\qquad\qquad + \; P(\{\textit{neither } k \textit{ nor } l \textit{ occur}\})\,.
\end{aligned}$$

3. (20) Assume that the random variables U and Z are independent with $U \sim \exp(1)$ and $Z \sim \mathrm{N}(0,1)$. Assume as known that for $a > 0$ and $b \geq 0$

$$\int_0^\infty \frac{1}{\sqrt{u}} e^{-au-\frac{b}{u}} du = \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}\,.$$

Define $X = \sqrt{2U}Z$.

a. (10) Show that the density of the random variable X is equal to

$$f_X(x) = \frac{1}{2} e^{-|x|}\,.$$

b. (10) Let X and Y be independent and equally distributed. Compute the density of the sum $S = X + Y$.

Hint: Use the formula

$$f_S(s) = \int_{-\infty}^{\infty} f_X(x) f_Y(s-x) dx\,.$$

4. (20) A red and a black die are rolled simultaneously and repeatedly. The rolls are independent and the outcomes on the two dice are independent. The dice are rolled until 6 appears on the black die. Denote by N the number of rolls needed until 6 appears including the last roll. Denote by X the biggest number of dots that appeared on the red die before N.

a. (10) Compute $P(X = k|N = n)$.

Hint: compute $P(X \leq k|N = n)$ *and note that*

$$P(X = k|N = n) = P(X \leq k|N = n) - P(X \leq k-1|N = n)\,.$$

b. (10) Compute $E(N|X = k)$. Assume as known that for $|a| < 1$ we have

$$\sum_{n=1}^{\infty} a^n = \frac{a}{1-a} \qquad \text{and} \qquad \sum_{n=1}^{\infty} n a^n = \frac{a}{(1-a)^2}\,.$$

5. (20) There are 5 different balls in an urn labelled with numbers 2, 4, 6, 8, 10. A ball is picked at random from the urn six times independently and with replacement. Denote by X the sum of the six numbers on the balls selected.

a. (10) Compute the generating function of the random variable X.

b. (10) Compute $P(X = 24)$.

6. (20) Slot machines have 3 reels with 7 different symbols each. When the lever is pulled, the reels stop independently of each other. The 7 symbols on each reel are equally likely to appear. The bet is always one unit. If all three symbols match the slot machine returns the bet and additional 4 units. If two symbols match the slot machine returns the bet and an additional unit. In every other case the bet is lost.

a. (10) Let X_1 denote the gambler's profit in one game. The profit can be 4 units, 1 unit or -1 unit. Compute $E(X_1)$ and $\mathrm{var}(X_1)$.

b. (10) Compute approximately the probability that after $n = 1000$ games the loss will be 150 units or less. Assume that the player has enough units to play 1000 games even if he loses every game.