University of Primorska

FAMNIT

Probability

Written examination

June 14th, 2017

Name and surname: ____________________ Identification number: | | | | | | | | |

Instructions

Read carefully the text of the problems before attempting to solve them. Five problems out of six count for 100%. You are allowed one A4 sheet with formulae and theorems. You have two hours.

Naloga	a.	b.	c.	d.	
1.			•	•	
2.			•	•	
3.			•	•	
4.			•	•	
5.			•	•	
6.			•	•	
Skupaj					

1. (20) Four students overslept and missed the exam. The next day they came to the professor with the excuse that they had had a flat tyre. The professor is forthcoming and gives them a make-up exam in two parts. The first part is worth 60% and consists of four exercises with the same number of points, and the second contains only the question "Which tyre was flat?" which is worth 40%. Each students solves each of the first four exercises with 70% probability, independently of other students and other exercises.

There is a catch, however: the students earn the 40% on the second part only if they all give the same answer to the question which of the four tyres was flat. Let us assume that the four students do not expect the question about the tyre and answer it independently of each other and independently of all other questions at random with all four tyres being equally likely.

a. (10) What is the probability that all four students pass the examination i. e. that all of them collect at least 50%?

b. (10) Given that all four students have passed the examination what is the conditional probability that they all gave the same answer to the question about the flat tyre?

2. (20) Let an urn contain balls of m different colors where $B_1, B_2, \ldots, B_m$ represent the numbers of balls of each colour. Let us select n balls from the urn at random and without replacement so that each sample of n balls is equally likely. Denote by $X_1, X_2, \ldots, X_m$ the numbers of balls of different colours.

a. (10) Compute $\mathrm{cov}(X_k, X_l)$ for $k \neq l$.

   Hint: what is the distribution of $X_k + X_l$*?*

b. (10) Assume that $B_1 = B_2 = \cdots = B_m = B$. Compute the probability that at least one of the colours is missing among the n balls selected. You do not need to simplify the sums obtained.

3. (20) Let the random vector (X, Y) have the density

$$f_{X,Y}(x, y) = \frac{1}{\pi^{3/2} \left(1 + x^2 + y^2\right)^{3/2}} .$$

a. (10) Compute the marginal densities $f_X(x)$ in $f_Y(y)$.

Hint: use the substitution $y = \sqrt{1 + x^2}\, u$.

b. (10) Let a and b be given numbers with $a^2 + b^2 = 1$. Compute the density of the vector

$$(U, V) = (aX - bY, bX + aY) .$$

4. (20) An urn contains $a \geq 1$ white and $b \geq 1$ black balls. In each round a ball is selected at random from the urn so that each ball has equal probability, and the choice is independent of previous selections. If the ball selected is white it is returned to the urn. If the ball selected is black it is not returned but instead a white ball is added to the urn.

a. (10) Let X_k be the number of white balls in the urn just after the k-th selection. Show that

$$E(X_k) = a + b - b\left(1 - \frac{1}{a+b}\right)^k .$$

Hint: compute $E(X_{k+1}|X_k = j)$.

b. (10) Compute the probability that a white ball is selected in the k-th round.

Hint: use the total probability formula for $H_l = \{X_{k-1} = l\}$.

5. (20) Let $X_1, X_2, \ldots$ be a sequence of non-negative integer valued random variables with values in $\{0, 1, \ldots, m\}$. Assume that

$$P\left(X_{n+1} = k-1 | X_n = k\right) = \frac{k}{m} \quad \text{and} \quad P\left(X_{n+1} = k+1 | X_n = k\right) = 1 - \frac{k}{m}\,.$$

for $0 \leq k \leq m$. Denote by $G_n(s)$ the probability generating function of the random variable X_n.

a. (10) Show that

$$G_{n+1}(s) = sG_n(s) + \frac{1-s^2}{m} \cdot G_n'(s)\,.$$

b. (10) Assume that the random variables X_1 and X_2 have the same distribution. Show that in this case

$$G_1(s) = G_2(s) = \left(\frac{1}{2} + \frac{s}{2}\right)^m\,.$$

Find the distribution of X_n for all $n \geq 1$.

6. (20) In a game of chance three fair dice are rolled. If no sixes show the player loses the stake of one euro. Otherwise he gets his stake back and in addition as many euros as there are sixes.

a. (10) What is the approximate probability that the House will have a loss after 1000 games?

b. (10) At least how many games must the game be played so that the House will have a profit of at least 1000 euros with 99% probability?