University of Primorska

FAMNIT

Probability

Written examination

August 16$^{\text{th}}$, 2017

Name and surname: ________________ Identification number: | | | | | | | | |

Instructions

Read carefully the text of the problems before attempting to solve them. Five problems out of six count for 100%. You are allowed one A4 sheet with formulae and theorems. You have two hours.

Problem	a.	b.	c.	d.	
1.			•	•	
2.			•	•	
3.			•	•	
4.			•	•	
5.			•	•	
6.			•	•	
Total					

1. (20) Assume one has 52 cards numbered $1, 2, \ldots, 13$, where every number appears exactly four times. The player is dealt exactly five cards from s well-shuffled deck of cards. Any seletion of five cards is assumed to be equally likely.

a. (10) Compute the probability that the player gets cards with five consecutive numbers, e.g., $\{3, 4, 5, 6, 7\}$. The order in which cards are dealt is irrelevant.

b. (10) Assume that a *Joker* card is added to the deck. The *Joker* card can replace any other card. There is 53 cards in the deck now. Example: if the *Joker* card is denoted by J, the sequence $\{3, \mathrm{J}, 5, 6, 7\}$ means five consecutive numbers because J replaces the number 4. Compute the probability that the player gets five consecutive numbers in this case.

2. (20) Two bored statisticians are independently rolling dice, one dice each, until the sum of their outcomes equals 6. Denote the number of rolls by X including the last roll.

a. (10) Find the distribution of random variable X and name it.

b. (10) Compute the probability that the sum of statisticians's outcomes will equal 6 after even number of throws.

3. (20) Let the random vector (X, Y) have the density

$$f_{X,Y}(x,y) = \frac{1}{4\pi\sqrt{1-\rho^2}}\, e^{-\frac{x^2+y^2}{2(1-\rho^2)}} \left(e^{\frac{\rho xy}{1-\rho^2}} + e^{-\frac{\rho xy}{1-\rho^2}}\right)$$

for $|\rho| < 1$ and $\rho \neq 0$.

a. (10) Compute the marginal densities. Are X and Y independent?

b. (10) Define $(U, V) = (X + Y, X - Y)$. Compute the density of (U, V) and the density of U.

4. (20) In the simple model of an epidemic one assumes that in the first wave of infections every individual in the population gets infected with probability p independently of all the others. In the second wave of infections every yet non-infected individual is infected with probability equal to the proportion of infected individuals in the first wave, independently of all the others. Let X be the number of infected individuals after the first wave of infection and by Y the number of all the infected individuals after the second wave of infection. More mathematically we have:

$$P\left(Y = l | X = k\right) = \binom{n-k}{l-k} \left(\frac{k}{n}\right)^{l-k} \left(1 - \frac{k}{n}\right)^{n-l}$$

for $0 \leq k \leq l \leq n$.

a. (10) Compute $E(Y|X = k)$.

b. (10) Compute $E(Y)$.

5. (20) Let the nonnegative random variable N have the generating function

$$G_N(s) = \frac{\log(1 - ps)}{\log(1 - p)}$$

for $p \in (0, 1)$. Assume as known that for $|x| < 1$

$$\log(1 - x) = -\sum_{k=1}^{n} \frac{x^k}{k} \,.$$

Let the random variables $X_1, X_2, \ldots$ be independent of N and of each other. Let $X = I_1 + \cdots + I_N$.

a. (10) Let $X_k \sim \text{Bernoulli}(\rho)$ for $k = 1, 2, \ldots$. Assume $p, \rho \in (0, 1)$. Find the distribution of the random variable X.

b. (10) Let $X_k \sim \text{Geom}\left(\frac{1}{2}\right)$. Find the distribution of X.

6. (20) Consider the following game: Players A and B will each toss a fair coin 1000 times. Denote by X the number of heads after 1000 tosses of the player A, and by Y the number of heads after 1000 tosses of the player B. If $|X - Y| \leq 15$, A wins, otherwise B wins.

a. (10) There are 4 possible outcomes if two fair coins are tossed independently: HH, HT, TH, TT. Every outcome happens with probability 1/4. Fill the missing part in the following sentence: The difference $X - Y$ equals the sum of ______ random numbers generated by selecting tickets at random with replacement from the box

$$\boxed{?}\quad \boxed{?}\quad \boxed{?}\quad \boxed{?}$$

Hint: What happens to the difference of the number of heads after a toss of two coins?

b. (10) Compute the approximate probability that the player A wins.