University of Primorska

FAMNIT

Probability

Written examination

June 29$^{\text{th}}$, 2017

Name and surname: ____________________ Identification number: | | | | | | | | |

Instructions

Read carefully the text of the problems before attempting to solve them. Five problems out of six count for 100%. You are allowed one A4 sheet with formulae and theorems. You have two hours.

Naloga	a.	b.	c.	d.	
1.			•	•	
2.			•	•	
3.			•	•	
4.			•	•	
5.			•	•	
6.			•	•	
Skupaj					

1. (20) n balls are thrown in r boxes. Assume the throws are independent and each box is hit with equal probability. Denote by X the number of empty boxes at the end.

a. (10) Define the events

$$A_k = \{k\text{-th box is empty}\}$$

for $k = 1, 2, \ldots, r$. Express the event $A = \{X = 0\}$ by events A_k.

b. (10) Compute $P(X = 0)$.

2. (20) A Pólya's urn contains b white and two red balls. On each step a ball is selected at random from the urn. Every time a ball of particular color is selected, it is returned to the urn and one more ball of the same colour is added to the urn. The selections continue until a red ball is selected. Assume that this happens with probability 1.

a. (10) Denote by N the number of selections until and including the one when a red ball is selected. Write down the distribution of this random variable.

b. (10) Compute $E(N)$.

   Hint: $E(N) = \sum_{n=1}^{\infty} P(N \geq n)$.

3. (20) The *Beta* distribution with parameters $p, q > 0$ is given by the density

$$f(x) = \frac{1}{\mathrm{B}(p,q)}\, x^{p-1}(1-x)^{q-1}$$

for $0 < x < 1$, where

$$\mathrm{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx\,.$$

a. (10) Let X and Y be independent and

$$X \sim \mathrm{Beta}\left(\frac{1}{2}, \frac{1}{2}\right) \quad \text{and} \quad Y \sim \mathrm{Beta}\left(1, \frac{1}{2}\right)\,.$$

Show that the density of the vector $(U, V) = (X, XY)$ is given by

$$f_{U,V}(u,v) = f_X(u)\, f_Y(v/u)\, \frac{1}{u}$$

for $0 < v < u < 1$. Compute the density XY explicitly. Assume as known

$$\int_v^1 \frac{1}{u}(1-u)^{-1/2}(u-v)^{-1/2}du = \frac{\pi}{\sqrt{v}}\,.$$

b. (10) Let X and Y be independent with

$$X \sim \mathrm{Beta}(a,b) \quad \text{and} \quad Y \sim \mathrm{Beta}(a+b,c)$$

for positive constants a, b and c. Compute the density of XY. Assume as known that

$$\int_v^1 \frac{(1-u)^{b-1}(u-v)^{c-1}du}{u^{b+c}} = \mathrm{B}(b,c)\, v^{-b}(1-v)^{b+c-1}\,.$$

4. (20) Random number generators generate strings of zeros and ones. Assume that the generated numbers are independent and that each generated number equals 1 with probability 1/2.

a. (10) When the quality of some generator of random numbers is tested, the random variable Y is defined which counts the numbers of appearances of two consecutive ones in the string of n random zeros and ones. The overlaipping occurencies are allowed in the sense that in 1011011110111 there are six appearances of two consecutive ones. Compute $E(Y)$.

b. (10) Let Z be the number of appearances of the sequence 011 in the set of n generated random numbers, where overlapping occurencies are not allowed. Compute $E(Z)$ and $\text{var}(Z)$.

5. (20) Assume that for the sequence of random variables $X_1, X_2, \ldots$ we have

$$E\left(s^{X_{n+1}} | X_n = k\right) = \left(\frac{1+s}{2}\right)^k \cdot e^{-\frac{\lambda}{2}(1-s)}$$

for $n \geq 1$ and $\lambda > 0$. Denote by $G_n(s)$ probability generating function of the random variable X_n.

a. (10) Let $X_1 \sim \text{Po}(\lambda)$. Find the distributions of $X_2, X_3, \ldots$.

b. (10) Assume that $X_1 \sim \text{Po}(\mu)$. Find the distributions of X_n for all $n \geq 1$.

6. (20) Berti opens a stand with a game involving three dice. In every game costs 1 euro and the three dice are rolled. If no sixes show Berti keeps the stake. If excactly one six shows, Berti gives returns the stake to the player with an additional 1 euro. If excactly two sixes show, Berti returns the stake to the player with additional 2 euros. If three sixes show, Berti returns the stake to the player with additional 14 euros. Assume the dice are fair and that all the rolls are independent.

a. (10) Compute the expected value and the variance of Berti's profit after n games.

b. (10) After approximately how many games will Berti have a positive profit with approximately 95% probability?