University of Primorska

FAMNIT, Mathematics

Probability

Midterm 2

June 4^{th}, 2018

Name and surname: ____________________ Identification number: | | | | | | | | |

Instructions

Read carefully the text of the problems before attempting to solve them. Five problems out of six count for 100%. You are allowed one A4 sheet with formulae and theorems. You have two hours.

Problem	a.	b.	c.	d.	
1.			•	•	
2.			•	•	
3.			•	•	
4.				•	
5.			•	•	
6.			•	•	
Total					

1. (20) Three numbers are randomly chosen from the set$\{1, 2, \ldots, n\}$ $(n \geq 3)$, so that every subset of three numbers is equally likely. Denote the smallest of the numbers by X, the middle one by Y and the largest one by Z.

a. (10) Compute the distribution of the difference $W = Z - X$.

Solution: the random variable W can take values $2, 3, \ldots, n-1$. For k from this set the probability of the event $\{W = k\}$ can be computed as the proportion of favourable outcomes. The number of all possible outcomes is $\binom{n}{3}$. We get a favorable outcome if X is a number from the set $\{1, 2, \ldots, n-k\}$, then Y is a number from the set $\{X+1, X+2, \ldots, X+k-1\}$ and $Z = X + k$. The number of favourable oucomes is $(n-k)(k-1)$ and it follows

$$P(W = k) = \frac{(n-k)(k-1)}{\binom{n}{3}} = \frac{6(n-k)(k-1)}{n(n-1)(n-2)}\,.$$

b. (10) Compute the distribution of the vector $(Y - X, Z - Y)$.

Solution: the possible values for the random vector $(Y - X, Z - Y)$ are pairs $(l, m) \in \mathbb{N}^2$ for which $l + m < n$. The favourable triples correspond to outcomes such that X is from the set $\{1, 2, \ldots, n-l-m\}$ and $Y = X + l$. We have

$$P(Y - X = l, Z - Y = m) = \frac{n-l-m}{\binom{n}{3}} = \frac{6(n-l-m)}{n(n-1)(n-2)}\,.$$

2. (20) Let $n \geq 2$ and let $X_1, X_2, \ldots, X_n$ be independent with

$$P(X_k = -1) = P(X_k = 1) = \tfrac{1}{2}$$

for all $k = 1, 2, \ldots, n$. Denote $S = X_1 + X_2 + \cdots + X_n$.

a. (10) Show that for every $k = 1, 2, \ldots, n$ the random variable $|S|$ is independent of X_k.

Hint: use symmetry.

Solution: by symmetry the random vectors $(X_1, \ldots, X_n)$ *and* $(-X_1, \ldots, -X_n)$ *have the same distribution. As a consequence* (X_1, S) *and* $(-X_1, -S)$ *have the same distribution. This in turn implies that* $(X_1, |S|)$ *and* $(-X_1, |S|)$ *have the same distribution and hence for* $s = 0, 1, \ldots, n$ *we have*

$$P(X_1 = 1, |S| = s) = P(X_1 = -1, |S| = s)\,.$$

On the other hand,

$$P(X_1 = 1, |S| = s) + P(X_1 = -1, |S| = s) = P(|S| = s)\,.$$

It follows that

$$P(X_1 = 1, |S| = s) = P(X_1 = -1, |S| = s) = \tfrac{1}{2} P(|S| = s)\,.$$

b. (10) Is $|S|$ independent of the random vector (X_1, X_2)?

Solution: no. The event $\{X_1 = 1, X_2 = -1, |S| = n\}$ *is impossible, whereas*

$$P(X_1 = 1, X_2 = -1)\, P(|S| = n) = 2^{-n-1} > 0\,.$$

3. (20) Let random variables X and Y have joint density equal to

$$f_{X,Y}(x,y) = e^{-x} \cdot \frac{1}{\sqrt{2\pi x}} e^{-\frac{(y-\theta x)^2}{2x}}$$

for $x > 0$ and $-\infty < y < \infty$, θ is given parameter.

a. (10) Show that the random variables X and

$$Z = \frac{Y - \theta X}{\sqrt{X}}$$

are independent.

Solution: we use the transformation formula. We define

$$(x,y) \xrightarrow{\Phi} \left(x, \frac{y-\theta x}{\sqrt{x}}\right),$$

that maps $(0,\infty) \times \mathbb{R}$ *into itself. We compute*

$$\Phi^{-1}(x,z) = (x, z\sqrt{x} + \theta x).$$

and

$$J_{\Phi^{-1}}(x,z) = \sqrt{x}.$$

It follows

$$f_{X,Z}(x,z) = f_{X,Y}(x, z\sqrt{x} + \theta x) \cdot |\sqrt{x}|$$

and

$$f_{X,Z}(x,z) = e^{-x} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}.$$

We notice that $X \sim \exp(1)$, $Z \sim N(0,1)$ *and that* X *and* Z *are independent.*

b. (10) Assume as known that for $a > 0$ and $b \geq 0$

$$\int_0^\infty \frac{1}{\sqrt{x}} e^{-ax-\frac{b}{x}} dx = \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}.$$

Compute the distribution of the random variable Y.

Solution: we compute the marginal density as

$$\begin{aligned} f_Y(y) &= \int_0^\infty e^{-x} \cdot \frac{1}{\sqrt{2\pi x}} e^{-\frac{(y-\theta x)^2}{2x}} dx \\ &= \frac{1}{\sqrt{2\pi}} e^{\theta} \int_0^\infty \frac{1}{\sqrt{x}} e^{-\frac{2+\theta^2 x}{2} - \frac{y^2}{2x}} dx \\ &= \frac{1}{\sqrt{2+\theta^2}} e^{\theta - \sqrt{(2+\theta^2)y^2}}. \end{aligned}$$

The known integral for $a = \frac{2+\theta^2}{2}$ *and* $b = \frac{y^2}{2}$ *is used.*

4. (20) There are $n \geq 5$ gamblers sitting around a round table. Each of them rolls his own die. All dice are standard (1 to 6 dots), fair (every number of dots has equal probability), and the rolls are independent. Define two numbers from the set $\{1, 2, 3, 4, 5, 6\}$ to be neighbours if their difference equals ± 1 or if the numbers are 1 and 6. Denote by W the number of gamblers whose two neighbours at the table have rolled a neighbouring number (for example, if a gambler rolls 1, his lefthand neighbour rolls 2 and his righthand neighbour rolls 6 then the gambler is counted).

a. (10) Compute $E(W)$ and $\mathrm{var}(W)$.

Solution: write $W = I_1 + I_2 + \cdots + I_n$, where the indicator I_i denotes the event that both neighbours of the i-th gambler roll neighbouring numbers of dots. The probability of this event is

$$E(I_i) = \frac{1}{9},$$

and by symmetry it follows

$$E(W) = \frac{n}{9}.$$

For the variance we can write

$$\mathrm{var}(W) = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{cov}(I_i, I_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} \left[E(I_i I_j) - E(I_i)E(I_j)\right].$$

For $i = j$ is $\mathrm{cov}(I_i, I_j) = 1/9 - 1/81 = 8/81$. If i-th and j-th gambler are neighbours, the covariance equals $\mathrm{cov}(I_i, I_j) = 1/27 - 1/81 = 2/81$. In all other cases $\mathrm{cov}(I_i, I_j) = 0$ because the events are independent. Summing up we get

$$\mathrm{var}(W) = \frac{8}{81} \cdot n + \frac{2}{81} \cdot 2n = \frac{4n}{27}.$$

b. (5) Let S be the number of gamblers rolling a six. Compute $\mathrm{cov}(W, S)$.

Solution: let $S = J_1 + J_2 + \cdots + J_n$ where te indicator J_j denotes the event, that the j-th gambler rolls a six. Indicators I_i and J_j are independent. Hence $\mathrm{cov}(I_i, J_j) = 0$ and consequently $\mathrm{cov}(W, S) = 0$.

c. (5) Are the random variables W and S independent?

Solution: no. We have $\{S = n\} \subseteq \{W = 0\}$ and $P(S = n, W = 0) = P(S = n) = 6^{-n}$ whereas $P(S = n)\, P(W = 0) < 6^{-n}$ since $P(W = 0) < 1$.

5. (20) Let $0 < a, b < 1$. Let the random variable $N \sim \text{Geom}(a)$. Let K be a random variable such that conditionally on $N = n$ its distribution is negative binomial $\text{NegBin}(n, b)$. More explicitly

- N is geometrically distributed $\text{Geom}(p)$, if for $n = 1, 2, 3, \ldots$ we have $P(N = n) = p(1-p)^{n-1}$.
- X has negative binomial distribution $\text{NegBin}(n, p)$ if for $k = n, n+1, n+2, \ldots$ we have $P(X = k) = \binom{k-1}{n-1} p^n (1-p)^{k-n}$.

a. (10) Determine the distribution of the random variable K.

Solution: by assumption for $k = n, n+1, n+2, \ldots$ we have

$$P(N = n) = a(1-a)^{n-1}\,; \qquad n = 1, 2, 3, \ldots$$
$$P(K = k | N = n) = \binom{k-1}{n-1} b^n (1-b)^{k-n}\,; \qquad k = n, n+1, n+2, \ldots$$

By the law of total probability

$$\begin{aligned} P(K = k) &= \sum_n P(N = n)\, P(K = k | N = n) \\ &= \sum_{n=1}^{k} \binom{k-1}{n-1} a(1-a)^{n-1} b^n (1-b)^{k-n} \\ &= ab \sum_{l=0}^{k-1} \binom{k-1}{l} (1-a)^l b^l (1-b)^{k-l-1} \\ &= ab(1-ab)^{k-1}\,. \end{aligned}$$

The random variable K has geometric distribution $\text{Geom}(ab)$.

b. (10) For all $k = 1, 2, 3, \ldots$ determine the conditional distribution of the random variable $N - 1$ given $K = k$.

Solution: conditionally on $K = k$ the random variable N can take values from the set $\{1, 2, \ldots k\}$. It follows that $N-1$ takes values from the set $\{0, 1, \ldots k-1\}$. For l from this last set we have

$$\begin{aligned} P(N - 1 = l | K = k) &= P(N = l + 1 | K = k) \\ &= \frac{P(N = l+1)\, P(K = n | N = l+1)}{P(K = k)} \\ &= \binom{k-1}{l} \frac{(1-a)^l b^l (1-b)^{k-l-1}}{(1-ab)^l} \\ &= \binom{k-1}{l} \left(\frac{(1-a)b}{1-ab}\right)^l \left(\frac{1-b}{1-ab}\right)^{k-1-l}\,. \end{aligned}$$

We conclude that the random variable $N-1$ *conditionally on* $K=k$ *has the binomial distribution* $\mathrm{Bin}\big(k-1, \frac{(1-a)b}{1-ab}\big)$.

6. (20) Slot machines have 3 reels with 7 symbols each. When the lever is pulled the reels stop independently of each other. The symbols on each reel appear equally likely. The bet is always one unit. If all three symbols match, the slot machine returns the bet and additional 4 units. If two symbols match, the slot machine returns the bet and additional 1 unit. In all other cases the bet is lost.

a. (10) Let X_1 denote the gambler's profit in one game which can be 4 units, 1 unit or -1 unit. Compute $E(X_1)$ and $\mathrm{var}(X_1)$.

Solution: let X_1 denote gambler's profit in the first game. We calculate

$$P(X_1 = 4) = 1/49, \quad P(X_1 = 1) = \frac{18}{49} \quad \textit{in} \quad P(X_1 = -1) = \frac{30}{49}\,.$$

It follows

$$E(X_1) = -\frac{8}{49} = -0,1633 \qquad \textit{and} \qquad \mathrm{var}(X_1) = \frac{64}{49} - \frac{64}{49^2} = 1,2795\,.$$

b. (10) Compute the approximate probability that after $n = 1000$ games the loss of the gambler will be 150 units or less. Assume that infinitely many units are available.

Solution: let $S_{1000} = X_1 + X_2 + \cdots + X_{1000}$. The central limit theorem gives

$$\begin{aligned} P(S_{1000} \geq -150) &= P\left(\frac{S_{1000} - 1000 \cdot E(X_1)}{\sqrt{1000 \cdot \mathrm{var}(X_1)}} \geq \frac{-150 - 1000 \cdot E(X_1)}{\sqrt{1000 \cdot \mathrm{var}(X_1)}}\right) \\ &\approx P(Z \geq 0,37) \\ &= 0.36\,. \end{aligned}$$