Name and surname: Identification number:

University of Primorska

FAMNIT, Mathematics

Probability

Midterm 1

April 16th, 2021

Instructions

Read carefully the text of the problems before attempting to solve them. Five problems out of six count for 100%. You are allowed one A4 sheet with formulae and theorems, and a handbook of mathematics. Time allowed: 120 minutes.

Question	a.	b.	c.	d.	Total
1.			•	•	
2.			•	•	
3.			•	•	
4.			•	•	
5.			•	•	
6.			•	•	
Total					

1. (20) There are r boxes. We toss n balls into the boxes so that tosses are independent and every box is hit with the same probability $1/r$. Assume that $n \geq r$. For $k = 1, 2, \ldots, r$, let A_k be the event that after n tosses there is at least one ball in the k-th box.

a. (10) Compute $P(A_1 \cup A_2 \cup \cdots \cup A_k)$.

Solution: the union is the event that there is at least one ball in the first k boxes. On each toss the probability that we hit one of the first k boxes is k/r. By independence

$$P(A_1^c \cap A_2^c \cap \cdots \cap A_k^c) = \left(1 - \frac{k}{r}\right)^n .$$

It follows that

$$P(A_1 \cup A_2 \cup \cdots \cup A_k) = 1 - \left(1 - \frac{k}{r}\right)^n .$$

b. (10) Compute $P(A_1 \cap A_2 \cap \cdots \cap A_r)$.

Hint: use the inclusion–exclusion formula.

Solution: passing to the opposite event and using symmetry, we compute

$$\begin{aligned} P(A_1^c \cup A_2^c \cup \cdots \cup A_r^c) &= \\ &= r\,P(A_1^c) - \binom{r}{2} P(A_1^c \cap A_2^c) + \binom{r}{3} P(A_1^c \cap A_2^c \cap A_3^c) - \\ &\qquad \cdots + (-1)^{r-1} P(A_1^c \cap \cdots \cap A_r^c) \\ &= \sum_{k=1}^{r} (-1)^{k-1} \binom{r}{k} \left(1 - \frac{k}{r}\right)^n . \end{aligned}$$

2. (20) A standard deck of cards contains 52 cards. There are 13 hearts, 13 spades, 13 clubs and 13 diamonds. The deck is shuffled well and each of the four players is dealt 13 cards.

a. (10) What is the probability that every player has cards of one suit only?

Solution: denote the event by A. There are

$$\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13} = \frac{52!}{(13!)^4}$$

equally likely ways to deal cards to four players. Out of these there are $4! = 24$ *ways such that every player has cards of one suit only. It follows*

$$P(A) = \frac{4! \cdot (13!)^4}{52!} .$$

b. (10) What is the conditional probability that the first player will have cards of one suit only given that **not** all players have cards of one suit only.

Solution: let $B = \{\textit{first player has cards of one suit only}\}$. *We need to compute*

$$P(B|A^c) = \frac{P(B \cap A^c)}{P(A^c)} = \frac{P(B) - P(A \cap B)}{1 - P(A)} = \frac{P(B) - P(A)}{1 - P(A)} .$$

We only need to compute $P(B)$. *The event* B *contains* $4 \cdot \frac{39!}{(13!)^3}$ *outcomes, hence*

$$P(B) = 4 \cdot \frac{13! \cdot 39!}{52!} = \frac{4}{\binom{52}{13}} .$$

The conditional probability equals

$$P(B|A^c) = \frac{4 \cdot 13! \left(39! - 3! \cdot (13!)^3\right)}{52! - 4! \cdot (13!)^4} .$$

3. (20) A player starts by rolling one die. The first time a six comes up he adds one die, and continues to roll two dice. When a double six comes up he adds a third die, and rolls the three dice until all three of them show a six. The process of adding a die each time all dice show a six is continued. Assume that all the dice and all the rolls are independent. Let X_m be the number of rolls in which strictly fewer than m dice are rolled. We assume that all six outcomes on the dice are equally likely.

a. (10) Find the distribution of X_3.

Hint: think about the event $\{X_2 = k, X_3 = n\}$.

Solution: the possible values for X_3 *are* $2, 3, 4\ldots$ *The random variable* X_3 *is the number of the roll when the third die is added. Let* $n = 2, 3, 4\ldots$ *If* $X_3 = n$ *then the possible values for* X_2 *are* $1, 2, \ldots, n-1$. *The event* $\{X_2 = k, X_3 = n\}$ *happens if there is no six in the first* $k-1$ *rolls of one die, a six on the* k*-th roll, and then* $n-k-1$ *rolls of two dice with no double sixes, and finally a double six. It follows that*

$$P(X_2 = k, X_3 = n) = \left(\frac{5}{6}\right)^{k-1} \cdot \frac{1}{6} \cdot \left(\frac{35}{36}\right)^{n-k-1} \cdot \frac{1}{36}\,.$$

Adding we get

$$\begin{aligned} P(X_3 = n) &= \sum_{k=1}^{n-1} P(X_2 = k, X_3 = n) \\ &= \frac{1}{6^3}\,\frac{\left(\frac{35}{36}\right)^{n-1} - \left(\frac{5}{6}\right)^{n-1}}{\frac{35}{36} - \frac{5}{6}} \\ &= \frac{1}{30}\left[\left(\frac{35}{36}\right)^{n-1} - \left(\frac{5}{6}\right)^{n-1}\right]. \end{aligned}$$

b. (10) Find the probability that $X_3 > X_2$.

Hint: condition on the number of rolls of one die.

Solution: let $A = \{X_3 > X_2\}$. *We have*

$$P(A \mid X_2 = k) = \left(\frac{35}{36}\right)^k .$$

Using the total probabilities formula we get

$$P(A) = \sum_{k=1}^{\infty} P(A \mid X_2 = k)\, P(X_2 = k) = \sum_{k=1}^{\infty} \left(\frac{35}{36}\right)^k \left(\frac{5}{6}\right)^{k-1} \cdot \frac{1}{6} = \frac{35}{41}\,.$$

4. (20) There are n chairs numbered counterclockwise 1 to n around a round table. The host seats b black and r red knights randomly around the table where $b + r = n$.

a. (10) Let A_k be the event that two knights flanking the knight on seat k are of different colours. Compute $P(A_k)$.

   Solution: by symmetry, the two knights flanking the knight on seat k are a randomly selected pair of knights. Of all $\binom{n}{2}$ pairs, there are br pairs of knights of different colours. It follows

$$P(A_k) = \frac{br}{\binom{n}{2}} = \frac{2br}{n(n-1)}\,.$$

b. (10) Let X be the number of knights who are flanked by knights of different colour. Compute $E(X)$.

   Solution: define indicators

$$I_k = \begin{cases} 1\,, & \textit{if the knight on seat } k \textit{ is flanked by knights of different colour;} \\ 0\,, & \textit{else;} \end{cases}$$

$$E(X) = \sum_{k=1}^{n} E(I_k) = \frac{2br}{n-1}\,.$$

5. (20) One of the tests for random number generators is the *Runs test.* We have m tickets with numbers $1, 2, \ldots, m$ on them. We choose tickets at random with replacement. The subsequent picks are independent, and all the tickets are equally likely. The selecting process is continued until the number on the chosen ticket is smaller or equal to the last number selected. Let X be the length of the run of strictly increasing numbers, and let Y be the number on the last ticket.

a. (15) List the possible values of the pair (X, Y), and compute $P(X = k, Y = l)$.

   Hint: striclty increasing strings of k numbers from the set $\{1, 2, \ldots, a\}$ are in a one-to-one correspondence with combinations of size k from that set.

   Solution: the possible pairs of values are (k, l) with $1 \leq l, k \leq m$. The event $\{X = k, Y = l\}$ happens if the last number in an increasing string of length k is greater or equal to l. There are $\binom{m}{k}$ increasing strings of k numbers. We need to subtract the number of strings containing the elements from $\{1, 2, \ldots, l-1\}$ of which there are $\binom{l-1}{k}$. Every string of $k+1$ values on tickets comes up with probability $\frac{1}{m^{k+1}}$ by independence. We have

$$P(X = k, Y = l) = \frac{1}{m^{k+1}} \left[\binom{m}{k} - \binom{l-1}{k} \right] .$$

   If $a < b$ we interpret $\binom{a}{b} = 0$.

b. (5) Find $P(Y = m)$.

   Solution: by the formula for marginal probabilities we have

$$P(Y = m) = \sum_{k=1}^{m} P(X = k, Y = m) = \sum_{k=1}^{m} \frac{1}{m^{k+1}} \left[\binom{m}{k} - \binom{m-1}{k} \right] .$$

   The binomial formula gives

$$P(Y = m) = \frac{1}{m} \left[\left(1 + \frac{1}{m}\right)^{m} - \left(1 + \frac{1}{m}\right)^{m-1} \right] = \frac{(m+1)^{m-1}}{m^{m+1}} .$$

6. (20) Two genes determine the eye colour in humans. A gene is either of type R or of type r. An individual has brown eyes if at least one gene is of type R, and has blue eyes else. The offspring inherits one gene from each of the parents where the gene to be transmitted is selected randomly from the two genes of the parent. The genes of the parents are equally likely, and the selections for the two parents are independent.

Assume that the proportions of men of types RR, Rr, rR and rr in the male population are p^2, $p(1-p)$, $(1-p)p$ in $(1-p)^2$, and the same for women. Assume that a new individual is created by independently selecting a man at random in the male population and a woman in the female population to be the parents.

a. (10) Denote $B = \{\text{offspring has brown eyes}\}$. Compute $P(B)$.

Hint: opposite event.

Solution: an individual inherits the gene of type r from the mother if she is of type Rr, rR or rr. The conditional probabilities of inheriting an r are $1/2$, $1/2$ and 1, hence

$$P(\textit{offspring inherits } r \textit{ from the mother}) = p(1-p) + (1-p)^2 = 1-p\,.$$

The probability for the father is the same. By independence

$$P(\textit{offspring is of type } rr) = (1-p)^2\,.$$

It follows that $P(B) = 1-(1-p)^2 = 2p-p^2$.

b. (10) Let $A = \{\text{parents have brown eyes}\}$ and $B = \{\text{offspring has brown eyes}\}$. Compute $P(A|B)$.

Hint: the events $H_{xy,zw} = \{$the mother is of type xy, the father is of type $zw\}$ for $x, y, z, w \in \{R, r\}$ are a partition, and we have

$$P(A\cap B) = \sum_{xy,zw} P(A\cap B\cap H_{xy,zw})\,.$$

Solution: we have

$$P(A\cap B) = \sum_{xy,zw} P(A\cap B\cap H_{xy,zw})\,.$$

The sum has 16 terms. If either $xy = rr$ or $zw = rr$, the probability of the intersection is 0. If $xy = RR$ and $zw = RR$, we have $P(A\cap B\cap H_{xx,yy}) = P(H_{xy,zw}) = p^4$. If $xy = RR$ and $zw = Rr$ or rR, we have $P(A\cap B\cap H_{xx,yy}) = P(H_{xy,zw}) = p^2\cdot p(1-p)$. Interchanging the roles of xy and zw gives us 4 such possibilities. We are left with

$$P(A\cap B\cap H_{Rr,Rr}) = P(B\cap H_{Rr,Rr}) = p^2(1-p)^2\cdot\frac{3}{4}\,.$$

Interchanging x and y, and z and w gives 4 cases. It follows

$$P(A\cap B) = p^4 + 4\cdot p^2(1-p)^2\cdot\frac{3}{4} + 4\cdot p^2\cdot p(1-p)\,.$$