University of Primorska

FAMNIT

Probability

Midterm 1

April 19$^{\text{th}}$, 2017

Name and surname: ______________________ Identification number: | | | | | | | | |

Instructions

Read carefully the text of the problems before attempting to solve them. Five problems out of six count for 100%. You are allowed one A4 sheet with formulae and theorems. You have two hours.

Problem	a.	b.	c.	d.	
1.			•	•	
2.				•	
3.				•	
4.			•	•	
5.			•	•	
6.				•	
Total					

1. (20) Suppose $m > 1$ begonias and $n > 1$ fuchsias are randomly arranged on a window sill. All orderings of the $m + n$ flowers are equally likely.

a. (10) What is the probability that to the right of the leftmost begonia there is a begonia?

   The complete solution should be in closed form, the allowed operations being addition, subtraction, multiplication and division.

   Hint: when counting arrangements, merge the two leftmost begonias.

   Solution: there are $\binom{m+n}{m}$ *possible arrangements of the flowers. We need to count the arrangements having a begonia to the right of the leftmost begonia. These are in bijective correspondence with the arrangements of* $m-1$ *begonias and* n *fuchsias: we get the desired arrangements in such a way that the two leftmost begonias are merged. The number of all such arrangements is* $\binom{m+n-1}{m-1}$*. The desired probability is* $\frac{m}{m+n}$*.*

b. (10) Let $k = 1, 2, \ldots, m + n$. What is the conditional probability that in the k-th position counted from the left there is a fuchsia given that to the right of the leftmost begonia there is a begonia?

   The complete solution should be in closed form, the allowed operations being addition, subtraction, multiplication, division and binomial symbols. Follow the convention that $\binom{a}{b} = 0$ for $a, b \in \mathbb{Z}$ and $b < 0$ or $b > a$, but still be carefull with boundary cases.

   Solution: we need to count the arrangements for which to the right of the leftmost begonia there is a begonia, and in the k*-th position there is a fuchsia. These arrangements (we will call them* originals*) are in bijective correspondence with the arrangements of* $m-1$ *begonias and* n *fuchsias (we will call these* adapted*):*

   - *In the original arrangement there are* k *fuchsias starting from the left and to the right of the leftmost begonia is also a begonia. These correspond to the adapted arrangements where there are* k *fuchsias starting from the left but there are* $m-1$ *begonias because we have merged two of them. There are* $\binom{m+n-k-1}{m-1}$ *such arrangements for all* k*.*
   - *In the original arrangements there is a fuchsia in the* k*-th position and the leftmost begonia is to the left of it, and there is a begonia to the right of the leftmost begonia. This corrensponds to the adapted arrangements for which there is a fuchsia in the* $(k-1)$*-st positon and at least one begonia to the left of the fuchsia, where we again reduce the number of begonias by 1. There are* $\binom{m+n-2}{m-1} - \binom{m+n-k}{m-1}$ *such arrangements when* $k \geq 2$*; for* $k = 1$ *there are no such arrangements.*

The conditional probability equals to

$$\frac{\binom{m+n-2}{m-1}}{\binom{m+n-1}{m-1}} = \frac{n}{m+n-1}\,,$$

for $k = 1$ *and for* $k = 2, 3, \ldots, m+n$ *equals to*

$$\frac{\binom{m+n-k-1}{m-1} + \binom{m+n-2}{m-1} - \binom{m+n-k}{m-1}}{\binom{m+n-1}{m-1}} = \frac{n}{m+n-1} - \frac{\binom{m+n-k-1}{m-2}}{\binom{m+n-1}{m-1}}\,.$$

2. (20) Two players A and B have an ordinary deck of 52 cards each. Both players shuffle their decks well and independently of each other. They both start placing cards on the table one by one face up simultaneously from the top of their respective decks. Use the inclusion/exclusion formula to compute the probabilities below.

a. (10) Let C be the event that at least once the players simultaneously place an Ace on the table. There are 4 Aces among 52 cards. Compute the probability $P(C)$. You do not need to simplify the resulting expressions.

Solution: define the events

$$C_i = \{\textit{the } i\textit{-th card that the two players simultaneously place are both Aces}\}$$

for $i = 1, 2, \ldots, 52$. *We have* $C = \cup_{i=1}^{52} C_i$. *We use the inclusion-exclusion formula, where we notice that the intersections of* 5 *or more events out of* $C_1, C_2, \ldots, C_n$ *are empty. Due to symmetry all the intersections of* k *different events have the same probability. It follows*

$$\begin{aligned} P(C) &= \binom{52}{1} P(C_1) - \binom{52}{2} P(C_1 \cap C_2) \\ &\quad + \binom{52}{3} P(C_1 \cap c_2 \cap C_3) - \binom{52}{4} P(C_1 \cap C_2 \cap C_3 \cap C_4)\,. \end{aligned}$$

By independence we have

$$P(C_1) = \left(\frac{4}{52}\right)^2,$$

$$P(C_1 \cap C_2) = \left(\frac{4 \cdot 3}{52 \cdot 51}\right)^2,$$

$$P(C_1 \cap C_2 \cap C_3) = \left(\frac{4 \cdot 3 \cdot 2}{52 \cdot 51 \cdot 50}\right)^2$$

and

$$P(C_1 \cap \cdots \cap C_4) = \left(\frac{4 \cdot 3 \cdot 2 \cdot 1}{52 \cdot 51 \cdot 50 \cdot 49}\right)^2.$$

All together we get

$$P(C) = \frac{15229}{54145} \doteq 0,281263\,.$$

b. (10) Let D be an event that the players at least once simultaneously place the same card on the table. Compute $P(D)$.

Solution: the problem is identical to the example from the lectures, where 52 *couples attend the ball, and upon leaving every woman chooses a man at random . From the lectures we have*

$$P(D) = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots - \frac{1}{52!}\,,$$

or

$$P(D) = \frac{33323980890946889067569406831865526501968231424164303372618082878 3}{52717761549636521942261854154512265996921245386198220800000000000000} \doteq 0{,}6321\,.$$

3. (20) An urn contains r white and r black balls. Assume balls are selected one by one at random without replacement.

a. (10) Let M be the number of balls selected until the first white ball is drawn including the white ball. Compute $P(M = k)$ for $k = 1, 2, \ldots, r + 1$.

   Solution: before the first white ball we need to draw black balls. It follows

$$P(M = k) = \frac{r}{2r} \cdot \frac{r-1}{2r-1} \cdots \frac{r-k+2}{2r-k+2} \cdot \frac{r}{2r-k+1}$$

b. (10) Let N be the number of balls selected until all r white or all r black balls have been drawn. Compute $P(N - k)$ for $k = r, r + 1, \ldots, 2r - 1$.

   Solution: the event $\{N = k\}$ happens, if the first $k - 1$ draws produce $r - 1$ balls of the same colour, and the last ball is of the same colour. The first $k - 1$ draws produce a random sample of all $2r$ balls. The probability that there are $k - 1$ white balls among them is by the hypergeometric distribution equal to

$$\frac{\binom{r}{r-1}\binom{r}{k-r}}{\binom{2r}{k-1}} .$$

   Conditionally on the above, the probability that the k-th ball will be white is $1/(2r - k + 1)$. By symmetry, the overall probability is

$$2 \cdot \frac{\binom{r}{r-1}\binom{r}{k-r}}{\binom{2r}{k-1}} \cdot \frac{1}{2r-k+1} .$$

4. (20) Let the density of the random variable X be given by

$$f_X(x) = \begin{cases} \frac{c}{x^{3/2}} e^{-\frac{1}{2x}} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

for some constant c.

a. (10) Show that for $x > 0$

$$F_X(x) = P(X \le x) = 2P\left(Z \ge \frac{1}{\sqrt{x}}\right),$$

where $Z \sim \mathrm{N}(0,1)$.

Solution: by definition

$$\begin{aligned} F_X(x) &= \int_0^x f_X(u)du \\ &= c\int_0^x \frac{1}{u^{3/2}} e^{-\frac{1}{2u}}\,du \\ &= 2c\int_{1/\sqrt{x}}^\infty e^{-\frac{v^2}{2}}\,dv \quad \textit{Substitution: } \tfrac{1}{u} = v^2 \\ &= 2c\sqrt{2\pi}\cdot\frac{1}{\sqrt{2\pi}}\int_{1/\sqrt{x}}^\infty e^{-\frac{v^2}{2}}\,dv \\ &= 2c\sqrt{2\pi}\left(1 - \Phi\left(\frac{1}{\sqrt{x}}\right)\right). \end{aligned}$$

The notation $\Phi(x)$ stands for the distribution function of standard normal distribution. We have $F_X(x) \to 1$ when $x \to \infty$. It follows

$$2c\sqrt{2\pi}\lim_{x\to\infty}\left(1 - \Phi\left(\frac{1}{\sqrt{x}}\right)\right) = c\sqrt{2\pi} = 1.$$

It follows that $c = \frac{1}{\sqrt{2\pi}}$.

b. (10) Let $Z \sim \mathrm{N}(0,1)$. What is the density of the random variable $Y = \frac{1}{Z^2}$? First compute $P(Y \le y)$ for $y > 0$.

Solution: we have

$$P(Y \le y) = P\left(\frac{1}{Z^2} \le y\right) = 2P\left(Z \ge \frac{1}{\sqrt{y}}\right) = 2\left(1 - \Phi\left(\frac{1}{\sqrt{y}}\right)\right).$$

It follows that $F_X(y) = F_Y(y)$, which means that X and Y have the same density.

5. (20) An urn contains b white and 3 red balls. Balls are selected from the urn at random without replacement. Let X be the number of white balls before the first red ball is drawn and Y the number of white balls between the first and the second red ball.

a. (10) Find the joint distribution of the random variables X and Y.

Solution: the possible values for the random variables are pairs (k, l), for which $k \geq 0$, $l \geq 0$ and $k + l \leq b$. The event $\{X = k, Y = l\}$ happens, if first k white balls is chosen, then a red ball, then l white ones and finally a red ball. Denote $b + 3 = n$. We compute

$$
\begin{aligned}
P(X = k, Y = l) &= \\
&= \frac{b(b-1)\cdots(b-k+1)}{n(n-1)\cdots(n-k+1)} \cdot \frac{3}{n-k} \cdot \\
&\qquad \cdot \frac{(b-k)(b-k-1)\cdots(b-k-l+1)}{(n-k-1)(n-k-2)\cdots(n-k-l)} \cdot \frac{2}{(n-k-l-1)} \\
&= \frac{b(b-1)\cdots(b-k-l+1)\cdot 3 \cdot 2}{n(n-1)\cdots(n-k-l-1)} \\
&= \frac{b! \cdot (n-k-l-2)! \cdot 3 \cdot 2}{(b-k-l)! \cdot n!} .
\end{aligned}
$$

b. (10) Show that the random variables X and Y have the same distribution. Compute the distribution of Y.

Hint: the distribution of X should be computed separately, not as marginal distribution. Then use the symmetry of the joint distribution.

Solution: the random variable X counts the number of white balls before the first red ball is drawn. The event $\{X = k\}$ happens if k white balls are chosen first and then a red ball. Denote $n = b + 3$. We get

$$
P(X = k) = \frac{b(b-1)\cdots(b-k+1)\cdot 3}{n(n-1)\cdots(n-k+1)(n-k)}
$$

for $k = 0, 1, \ldots, b$. On the other hand, we can get the distributions of X and Y as marginal distributions of the joint distribution. Because it is a symmetric function of k and l, the distributions of X and Y are equal.

6. (20) Moe, Curly and Larry are playing a game in which any of them can win. In every round Moe wins with probability a, Curly wins with probability b and Larry wins with probability c ($a + b + c = 1$). The rounds are independent of each other. They are playing until one player wins twice in two or three consecutive rounds. The winner of the last round is the winner of the three-player competition.

a. (15) What is the probability that Moe wins the competition in such a way that he wins the last two games?

A full answer should be in closed form. The allowed operations are addition, subtraction, multiplication and division.

Hints:

- *Proceed backwards from the last three rounds.*
- *Treat the case where the last two rounds are the only rounds played separately.*
- *Suppose that four rounds are played and that the the winner of the third round from the back is Larry. Who is the winner of the fourth round from the back?*

Solution: Moe wins in the described way in the following series of wins:

$$\begin{array}{cc} \multicolumn{2}{c}{MM} \\ L\overline{MCL}MM & C\overline{MLC}MM \\ CL\overline{MCL}MM & LC\overline{MLC}MM \\ MCL\overline{MCL}MM & MLC\overline{MLC}MM \end{array}$$

Here the overline over a sequence means that this sequence can appear zero times, one time or multiple times. The desired probability equals:

$$\begin{aligned} &a^2\left[1 + (c + bc + abc + b + bc + abc)\sum_{n=0}^{\infty}(abc)^n\right] \\ &= a^2\left[1 + \frac{b + c + 2bc + 2abc}{1 - abc}\right] \\ &= \frac{a^2\big(1 + b + c + 2bc + abc\big)}{1 - abc} \\ &= \frac{a^2(2 - a + 2bc + abc)}{1 - abc}\,. \end{aligned}$$

b. (5) What is the probability that Moe wins the competition?

A full answer should be in closed form. The allowed operations are addition, subtraction, multiplication and division.

Solution: in addition to the first part, Moe can win in any of the following ways:

$$\begin{array}{rr} \overline{MCL}MCM & \overline{MLC}MLM \\ L\overline{MCL}MCM & C\overline{MLC}MLM \\ CL\overline{MCL}MCM & LC\overline{MLC}MLM \end{array}$$

All the ways of winning are disjoint and hence

$$\begin{aligned} & a^2\left[\frac{2-a+2bc+abc}{1-abc} + \Big(b(1+c+bc)+c(1+b+bc)\Big)\sum_{n=0}^{\infty}(abc)^n\right] \\ & \quad = \frac{a^2\big(2-a+b+c+bc(4+a+b+c)\big)}{1-abc} \\ & \quad = \frac{a^2(3-2a+5bc)}{1-abc}\,. \end{aligned}$$