University of Primorska

FAMNIT, Mathematics

Probability

Midterm 1

April 25$^{\text{th}}$, 2018

Name and surname: ____________________ Identification number: ☐☐☐☐☐☐☐☐

## Instructions

Read carefully the text of the problems before attempting to solve them. Five problems out of six count for 100%. You are allowed one A4 sheet with formulae and theorems. You have two hours.

| Problem | a. | b. | c. | d. | |
|---|---|---|---|---|---|
| 1. | | | • | • | |
| 2. | | | | | |
| 3. | | | • | • | |
| 4. | | | • | • | |
| 5. | | | • | • | |
| 6. | | | | • | |
| Total | | | | | |

**1.** (20) A fair die is rolled $n$ times. The rolls are numbered by $1, 2, \ldots, n$ and are assumed independent. For $k = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, n$ denote by $A_k$ the event that the $k$-th roll is the first one to show one dot, and by $B_l$ the event that the $l$-th roll is the last one to show six dots.

a. (10) Compute $P(A_k)$ and $P(B_l)$ for every $k$ and $l$.

*Solution: the event $A_k$ happens if in the first $k-1$ rolls there are no dots, the first dot is rolled on the $k$-th roll; it follows that $P(A_k) = \left(\frac{5}{6}\right)^{k-1} \cdot \frac{1}{6}$. The event $B_l$ happens if in the last $n-l$ rolls there is no six but it is comes up on $l$-th roll; that means that $P(B_l) = \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{n-l}$.*

b. (10) Determine for which $k$ and $l$ the events $A_k$ and $B_l$ are independent.

*Solution: we distinguish three cases:*

- *For $k < l$ the event $A_k \cap B_l$ means that in first $k-1$ rolls there no dots, the $k$-th roll is a dot, on the $l$-th roll six is rolled and on last $n-l$ rolls no sixes are rolled. That means $P(A_k \cap B_l) = \left(\frac{5}{6}\right)^{k-1} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{n-l}$, which equals $P(A_k)\, P(B_l)$, which implies that the events $A_k$ and $B_l$ are independent.*
- *For $k > l$ the event $A_k \cap B_l$ means that on the first $l-1$ rolls no dots are rolled, on the $l$-th roll a six is rolled, and between no dots or sixes are rolled, on the $k$-th roll a dot is rolled, and on last $n-k$ rolls no sixes are rolled. It follows $P(A_k \cap B_l) = \left(\frac{5}{6}\right)^{n-k+l-1} \left(\frac{2}{3}\right)^{k-l-1} \left(\frac{1}{6}\right)^2$, which is not equal to $P(A_k)\, P(B_l)$, so in this case the events $A_k$ and $B_l$ are dependent.*
- *For $k = l$ the event $A_k \cap B_l$ is impossible, so $P(A_k \cap B_l) = 0$, which means that $A_k$ and $B_l$ are dependent events.*

**2.** (20) For dinner, $2n$ seats are arranged around a round table for $n$ couples to be seated. The host will seat them in such a way that men and women will alternate but otherwise at random. You can imagine that the seats are numbered counterclockwise by 1 to $2n$. Women will be seated on seats $1, 3, \ldots, 2n-1$, and men will be seated on seats $2, 4, \ldots, 2n$, both at random independently of each other. We would like to find the probability that nobody sits next to his or her partner. Denote by

$$A_i = \{\text{seats } i \text{ and } i+1 \text{ are occupied by one of the couples}\}$$

where $2n+1$ is interpreted as 1.

a. (5) Compute $P(A_i)$ for all $i = 1, 2 \ldots, 2n$.

*Solution: the spouse of the person who is seated on the $i$-th seat occupies any of the $n$ seats with equal probability (conditional on knowing who is in the $i$-th seat). It follows that*

$$P(A_i) = \frac{1}{n}\,.$$

b. (5) Compute $P(A_i \cap A_j)$.

*Solution: if $i$ and $j$ are neighbouring seats, the intersection is an impossible event. If the seats $i$ and $j$ are not neighbouring seats, from each of the pairs $\{i, i+1\}$ and $\{j, j+1\}$ there must be a seat for a woman. The husbands can be then seated in $n(n-1)$ equally likely ways, out of which just one is favourable. It follows*

$$P(A_i \cap A_j) = \frac{1}{n(n-1)}\,.$$

c. In which cases is the probability $P(A_{i_1} \cap \cdots \cap A_{i_k})$ different from zero? For those cases compute the probability of the intersection.

*Solution: without loss of generality, we can assume that indices $i_1, i_2, \ldots, i_k$ are distinct. Then the probability of the intersection of the events is different from 0 zero if and only if the pairs of neighbour seats $\{i_1, i_1+1\}, \{i_2, i_2+1\}, \ldots, \{i_k, i_k+1\}$ are non-overlapping. The probability is computed as in the case b.: from every pair $\{i_j, i_j+1\}$ one of the seats belong to a woman. The husbands can be seated on $n(n-1)\cdots(n-k+1)$ equally likely ways and only one is favourable. It follows*

$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = \frac{1}{n(n-1)\cdots(n-k+1)} = \frac{(n-k)!}{n!}\,.$$

d. (5) Assume as known that one can choose $k$ non-overlapping pairs of adjacent seats among $2n$ seats around the table in

$$S_{k,n} = \binom{2n-k}{k} \frac{2n}{2n-k}$$

ways. Compute the probability that no two partners will sit together. You do not need to simplify the sums.

*Solution: let $A$ be event that no couple are seated together. Using inclusion exclusion formula one gets*

$$\begin{aligned} P(A) &= 1 - P\left(\bigcup_{i=1}^{2n} A_i\right) \\ &= 1 - \sum_{k=1}^{n} (-1)^{k-1} S_{k,n} \cdot \frac{(n-k)!}{n!} \\ &= \sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} \frac{2n}{2n-k} \cdot \frac{(n-k)!}{n!} . \end{aligned}$$

**3.** (20) Two cautious robbers A and B decide that they will go to "work" alternately until one of them is caught in the act. Assume that the outcomes of robbeires are independent of each other. Robber A is caught with probability $a$, and robber B with probability $b$. Robber A goes to work the first night.

a. (10) What is the probability that robber A will be caught before robber B?

*Solution: denote*

$$A_k = \{A \text{ gets caught the } k\text{-th night, before that the robberies are successfull}\}.$$

*This event happens when A is successfull $(k-1)$-times and B is successfull $(k-1)$-times and then A gets caught. Because the outcomes of the robberies are independent of each other, we have*

$$P(A_k) = (1-a)^{k-1}(1-b)^{k-1}a\,.$$

*Events $A_k$ are disjoint for $k = 1, 2, \ldots$, and their union is event that A gets caught first.*

$$\begin{aligned} &P(A \text{ gets caught before } B) \\ &= \sum_{k=1}^{\infty} P(A_k) \\ &= a\sum_{k=1}^{\infty} \big((1-a)(1-b)\big)^{k-1} \\ &= a\,\frac{1}{1-(1-a)(1-b)} \\ &= \frac{a}{a+b-ab}\,. \end{aligned}$$

b. (10) Let $X$ be the number of robberies until one of the robbers is caught including the last unsuccessful robbery. Compute the distribution of the random variable $X$.

*Solution: the possible values for random variable $X$ are $n = 1, 2, 3, \ldots$ We have to distinguish between even and odd $n$. Let $n = 2k$. In this case, A is successful $k$-times, B is successfull $(k-1)$-times and then B gets caught. It follows*

$$P(X = 2k) = (1-a)^k(1-b)^{k-1}b\,.$$

*For $n = 2k-1$:*

$$P(n = 2k-1) = (1-a)^{k-1}(1-b)^{k-1}a\,.$$

**4.** (20) Let the random variable $X$ have the Weibull density given by

$$f_X(x) = \frac{\alpha}{\sigma}\left(\frac{x}{\sigma}\right)^{\alpha-1} e^{-\left(\frac{x}{\sigma}\right)^{\alpha}}$$

for $x > 0$ and 0 otherwise, with $\alpha, \sigma > 0$.

a. (10) Find the density of the random variable

$$Y = \left(\frac{X}{\sigma}\right)^{\alpha}.$$

*Solution: notice that*

$$F_X(x) = 1 - e^{-\left(\frac{x}{\sigma}\right)^{\alpha}}$$

*for $x > 0$. Compute for $y > 0$*

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P\left(\left(\frac{X}{\sigma}\right)^{\alpha} \leq y\right) \\ &= P\left(X \leq \sigma y^{1/\alpha}\right) \\ &= 1 - e^{-\left(\frac{\sigma y^{1/\alpha}}{\sigma}\right)^{\alpha}} \\ &= 1 - e^{-y}. \end{aligned}$$

*It follows*

$$f_Y(y) = e^{-y}$$

*for $y > 0$ and 0 otherwise, or $Y \sim \exp(1)$.*

b. (10) Let $U \sim U(0,1)$. Show that the random variable

$$Z = \sigma\left(-\log U\right)^{1/\alpha}$$

has the Weibull density with parameters $\alpha, \sigma$.

*Solution: the random variable $Z$ has positive values. For $z > 0$ compute*

$$\begin{aligned} P(Z \leq z) &= P\left(\sigma\left(-\log U\right)^{1/\alpha} \leq z\right) \\ &= P\left(-\log U \leq \left(\frac{z}{\sigma}\right)^{\alpha}\right) \\ &= P\left(\log U \geq -\left(\frac{z}{\sigma}\right)^{\alpha}\right) \\ &= P\left(U \geq e^{-\left(\frac{z}{\sigma}\right)^{\alpha}}\right) \\ &= 1 - e^{-\left(\frac{z}{\sigma}\right)^{\alpha}}. \end{aligned}$$

**5.** (20) (20) Let the random variable $X$ have the distribution

$$P(X = k) = \binom{2n-k}{n}\left(\frac{1}{2}\right)^{2n-k}$$

for $k = 0, 1, \ldots, n$.

a. (10) Show that

$$\sum_{k=0}^{n}(2n-k+1)P(X=k) = 2(n+1)\left(1-\binom{2n+2}{n+1}\left(\frac{1}{2}\right)^{2(n+1)}\right).$$

*Hint: if one takes $n+1$ instead of $n$, the probabilities in the distribution still sum up to 1. Check that*

$$(2n-k+1)\binom{2n-k}{n} = (n+1)\binom{2(n+1)-(k+1)}{n+1}.$$

*Solution: compute*

$$\begin{aligned}
&\sum_{k=0}^{n}(2n-k+1)P(X=k) \\
&= \sum_{k=0}^{n}(2n-k+1)\binom{2n-k}{n}\left(\frac{1}{2}\right)^{2n-k} \\
&= \sum_{k=0}^{n}2(n+1)\binom{2(n+1)-(k+1)}{n+1}\left(\frac{1}{2}\right)^{2(n+1)-(k+1)} \\
&= 2(n+1)\left(1-\binom{2n+2}{n+1}\left(\frac{1}{2}\right)^{2(n+1)}\right).
\end{aligned}$$

b. (10) Compute $E(X)$.

*Solution: from the first part follows*

$$2n+1-E(X) = 2(n+1)\left(1-\binom{2n+2}{n+1}\left(\frac{1}{2}\right)^{2(n+1)}\right).$$

*It follows*

$$E(X) = -1+2(n+1)\binom{2n+2}{n+1}\left(\frac{1}{2}\right)^{2n+2}.$$

*The result can be written as*

$$E(X) = -1 + (2n+1)\binom{2n}{n}\left(\frac{1}{2}\right)^{2n} .$$

**6.** (20) We toss $n$ balls in $r$ boxes where $n \geq r$. Assume the tosses are independent and we hit each box with the same probability $p$. Denote by $X$ the number of empty boxes at the end.

a. (5) Denote $P(X = 0) = b(n, r)$ and compute this probability.

*Hint: inclusions and exclusions.*

*Solution: denote* $A = \{X = 0\}$ *and define*

$$A_k = \{k\text{-th box is empty}\}$$

*for* $k = 1, 2, \ldots, r$. *We have* $A^c = \cup_{k=1}^{r} A_k$. *By the inclusion-exclusion formula we need probabilities* $P(A_1 \cap \cdots \cap A_k)$ *for all* $k$. *In other words, we compute probability that on every toss* $r - k$ *boxes are hit. The throws are independent and it follows*

$$P(A_1 \cap \cdots \cap A_k) = \left(\frac{r-k}{r}\right)^n .$$

*Because of symmetry, the intersection of any* $k$ *events among* $A_1, \ldots, A_r$ *has the same probability, and it follows*

$$P(A) = 1 - P(A^c) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} \left(\frac{r-k}{r}\right)^n .$$

b. (10) Compute the distributon of $X$.

*Hint: what is the probability of* $\{X = k\}$, *conditionally on the event that a given assortment of* $k$ *boxes are exactly the empty ones.*

*Solution: for fixed* $k = 0, 1, \ldots, r-1$, *the empty* $k$ *boxes can be chosen in* $\binom{r}{k}$ *ways. Conditional on this, the balls must be tossed into the other* $r - k$ *boxes. It follows*

$$P(X = k) = \binom{r}{k} \left(\frac{r-k}{r}\right)^n b(n, r-k) =$$

$$= \sum_{l=0}^{r-k} (-1)^l \frac{r!}{k!\, l!\, (r-k-l)!} \left(\frac{r-k-l}{r}\right)^n .$$

c. (5) Compute $E(X)$.

*Solution: Define*

$$I_k = \begin{cases} 1, & \text{if } k\text{-th box is empty} \\ 0 & \text{otherwise.} \end{cases}$$

*We have* $X = I_1 + \cdots + I_r$ *and the indicators have the same distribution due to symmetry. It follows*

$$E(X) = r\,P(I_1 = 1) = r\left(\frac{r-1}{r}\right)^n .$$